Samacheer Kalvi Maths

9^{th} Std

Remark If a is a rational number and $\sqrt{b}$ is an irrational number then

(i) $a+\sqrt{b}$ is irrational (ii) $a-\sqrt{b}$ is irrational

(iii) $a\sqrt{b}$ is irrational (iv) $\frac{a}{\sqrt{b}}$ is irrational (v) $\frac{\sqrt{b}}{a}$ is irrational

For example, (i) $2+\sqrt{3}$ is irrational (ii) $2-\sqrt{3}$ is irrational

(iii) $2\sqrt{3}$ is irrational (iv) $\frac{2}{\sqrt{3}}$ is irrational

2.4.4 Square Root of Real Numbers

Let $a > 0$ be a real number. Then $\sqrt{a} = b$ means $b^2 = a$ and $b > 0$.

2 is a square root of 4 because $2 \times 2 = 4$, but -2 is also a square root of 4 because $(-2)\times(-2) = 4$. To avoid confusion between these two we define the symbol $\sqrt{\ }$, to mean the principal or positive square root.

Let us now mention some useful identities relating to square roots.

Let a and b be positive real numbers. Then	
1	$\sqrt{ab} = \sqrt{a}\sqrt{b}$
2	$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
3	$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$
4	$(a+\sqrt{b})(a-\sqrt{b}) = a^2-b$
5	$(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{d}) = \sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$
6	$(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab}$

1.2 Algebraic Identities

Key Concept **Algebraic Identities**

An identity is an equality that remains true regardless of the values of any variables that appear within it.

We have learnt the following identities in class VIII. Using these identities let us solve some problems and extend the identities to trinomials and third degree expansions.

$(a+b)^2 \equiv a^2+2ab+b^2$ $\qquad$ $(a+b)(a-b) \equiv a^2-b^2$

$(a-b)^2 \equiv a^2-2ab+b^2$ $\qquad$ $(x+a)(x+b) \equiv x^2+(a+b)x+ab$

Example 1.1

Expand the following using identities

(i) $(2a+3b)^2$ (ii) $(3x-4y)^2$ (iii) $(4x+5y)(4x-5y)$ (iv) $(y+7)(y+5)$

Solution

(i) $(2a+3b)^2 = (2a)^2+2(2a)(3b)+(3b)^2$

$= 4a^2+12ab+9b^2$

(ii) $(3x-4y)^2 = (3x)^2-2(3x)(4y)+(4y)^2$

$= 9x^2-24xy+16y^2$

(iii) $(4x+5y)(4x-5y) = (4x)^2-(5y)^2$

$= 16x^2-25y^2$

(iv) $(y+7)(y+5) = y^2+(7+5)y+(7)(5)$

$= y^2+12y+35$

1.2.1 Expansion of the Trinomial $(x \pm y \pm z)^2$

$$\begin{aligned}(x+y+z)^2 &= (x+y+z)(x+y+z)\\ &= x(x+y+z)+y(x+y+z)+z(x+y+z)\\ &= x^2+xy+xz+yx+y^2+yz+zx+zy+z^2\\ &= x^2+y^2+z^2+2xy+2yz+2zx\end{aligned}$$

$$(x+y+z)^2 \equiv x^2+y^2+z^2+2xy+2yz+2zx$$

(ii) $(x-y+z)^2 = [x+(-y)+z]^2$

$= x^2+(-y)^2+z^2+2(x)(-y)+2(-y)(z)+2(z)(x)$

$$= x^2 + y^2 + z^2 - 2xy - 2yz + 2zx$$

$$(x - y + z)^2 \equiv x^2 + y^2 + z^2 - 2xy - 2yz + 2zx$$

In the same manner we get the expansion for the following

(iii) $(x + y - z)^2 \equiv x^2 + y^2 + z^2 + 2xy - 2yz - 2zx$

(iv) $(x - y - z)^2 \equiv x^2 + y^2 + z^2 - 2xy + 2yz - 2zx$

Example 1.2

Expand (i) $(2x + 3y + 5z)^2$ (ii) $(3a - 7b + 4c)^2$ (iii) $(3p + 5q - 2r)^2$

(iv) $(7l - 9m - 6n)^2$

Solution

(i) $(2x + 3y + 5z)^2 = (2x)^2 + (3y)^2 + (5z)^2 + 2(2x)(3y) + 2(3y)(5z) + 2(5z)(2x)$

$= 4x^2 + 9y^2 + 25z^2 + 12xy + 30yz + 20zx$

(ii) $(3a - 7b + 4c)^2$

$= (3a)^2 + (-7b)^2 + (4c)^2 + 2(3a)(-7b) + 2(-7b)(4c) + 2(4c)(3a)$

$= 9a^2 + 49b^2 + 16c^2 - 42ab - 56bc + 24ca$

(iii) $(3p + 5q - 2r)^2$

$= (3p)^2 + (5q)^2 + (-2r)^2 + 2(3p)(5q) + 2(5q)(-2r) + 2(-2r)(3p)$

$= 9p^2 + 25q^2 + 4r^2 + 30pq - 20qr - 12rp$

(iv) $(7l - 9m - 6n)^2$

$= (7l)^2 + (-9m)^2 + (-6n)^2 + 2(7l)(-9m) + 2(-9m)(-6n) + 2(-6n)(7l)$

$= 49l^2 + 81m^2 + 36n^2 - 126lm + 108mn - 84nl$

1.2.2 Identities Involving Product of Binomials $(x+a)(x+b)(x+c)$

$$\begin{aligned}(x + a)(x + b)(x + c) &= [(x+a)(x+b)](x+c)\\ &= [x^2 + (a+b)x + ab](x+c)\\ &= x^3 + (a+b)x^2 + abx + cx^2 + c(a+b)x + abc\\ &= x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc\end{aligned}$$

$$\boxed{(x+a)(x+b)(x+c) \equiv x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc}$$

1.2.3 Expansion of $(x \pm y)^3$

In the above identity by substituting $a = b = c = y$, we get

$$(x+y)(x+y)(x+y) = x^3 + (y+y+y)x^2 + [(y)(y)+(y)(y)+(y)(y)]x + (y)(y)(y)$$

$$(x+y)^3 = x^3 + (3y)x^2 + (3y^2)x + y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$$

(or) $$(x+y)^3 \equiv x^3 + y^3 + 3xy(x+y)$$

Replacing y by $-y$ in the above identity, we get

$$(x-y)^3 \equiv x^3 - 3x^2y + 3xy^2 - y^3$$

(or) $$(x-y)^3 \equiv x^3 - y^3 - 3xy(x-y)$$

Using these identities of 1.2.2 and 1.2.3, let us solve the following problems.

Example 1.3

Find the product of

(i) $(x+2)(x+5)(x+7)$ (ii) $(a-3)(a-5)(a-7)$ (iii) $(2a-5)(2a+5)(2a-3)$

Solution

(i) $(x+2)(x+5)(x+7)$

$= x^3 + (2+5+7)x^2 + [(2)(5)+(5)(7)+(7)(2)]x + (2)(5)(7)$

$= x^3 + 14x^2 + (10+35+14)x + 70$

$= x^3 + 14x^2 + 59x + 70$

(ii) $(a-3)(a-5)(a-7) = [a+(-3)][a+(-5)][a+(-7)]$

$= a^3 + (-3-5-7)a^2 + [(-3)(-5)+(-5)(-7)+(-7)(-3)]a + (-3)(-5)(-7)$

$= a^3 - 15a^2 + (15+35+21)a - 105$

$= a^3 - 15a^2 + 71a - 105$

(iii) $(2a-5)(2a+5)(2a-3) = [2a+(-5)][2a+5][2a+(-3)]$

$= (2a)^3 + (-5+5-3)(2a)^2 + [(-5)(5)+(5)(-3)+(-3)(-5)](2a) + (-5)(5)(-3)$

$= 8a^3 + (-3)4a^2 + (-25-15+15)2a + 75$

$= 8a^3 - 12a^2 - 50a + 75$

Example 1.4

If $a+b+c = 15$, $ab+bc+ca = 25$ find $a^2+b^2+c^2$.

Solution We have $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$. So,

$15^2 = a^2+b^2+c^2+2(25)$

$225 = a^2+b^2+c^2+50$

$\therefore a^2+b^2+c^2 = 225-50 = 175$

Example 1.5

Expand (i) $(3a+4b)^3$ (ii) $(2x-3y)^3$

Solution

(i) $(3a+4b)^3 = (3a)^3 + 3(3a)^2(4b) + 3(3a)(4b)^2 + (4b)^3$

$= 27a^3 + 108a^2b + 144ab^2 + 64b^3$

(ii) $(2x-3y)^3 = (2x)^3 - 3(2x)^2(3y) + 3(2x)(3y)^2 - (3y)^3$

$= 8x^3 - 36x^2y + 54xy^2 - 27y^3$

Example 1.6

Evaluate each of the following using suitable identities.

(i) $(105)^3$ (ii) $(999)^3$

Solution

(i) $(105)^3 = (100+5)^3$

$= (100)^3 + (5)^3 + 3(100)(5)(100+5)$ $(\because (x+y)^3 = x^3+y^3+3xy(x+y))$

$= 1000000 + 125 + 1500(105)$

$= 1000000 + 125 + 157500 = 1157625$

(ii) $(999)^3 = (1000-1)^3$

$= (1000)^3 - (1)^3 - 3(1000)(1)(1000-1)$

$(\because (x-y)^3 = x^3 - y^3 - 3xy(x-y))$

$= 1000000000 - 1 - 3000(999)$

$= 1000000000 - 1 - 2997000 = 997002999$

Some Useful Identities involving sum ,difference and product of x and y

$$x^3 + y^3 \equiv (x+y)^3 - 3xy(x+y)$$
$$x^3 - y^3 \equiv (x-y)^3 + 3xy(x-y)$$

Let us solve some problems involving above identities.

Example 1.7

Find $x^3 + y^3$ if $x + y = 4$ and $xy = 5$

Solution We know that $x^3 + y^3 = (x+y)^3 - 3xy(x+y)$

$\therefore \quad x^3 + y^3 = (4)^3 - 3(5)(4) = 64 - 60 = 4$

Example 1.8

Find $x^3 - y^3$ if $x - y = 5$ and $xy = 16$

Solution We know that $x^3 - y^3 = (x-y)^3 + 3xy(x-y)$

$\therefore \quad x^3 - y^3 = (5)^3 + 3(16)(5) = 125 + 240 = 365$

Example 1.9

If $x + \frac{1}{x} = 5$, find the value of $x^3 + \frac{1}{x^3}$

Solution We know that $x^3 + y^3 = (x+y)^3 - 3xy(x+y)$

Put $y = \frac{1}{x}$, $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$

$= (5)^3 - 3(5) = 125 - 15 = 110$

Example 1.10

If $y - \frac{1}{y} = 9$, find the value of $y^3 - \frac{1}{y^3}$

Solution We know that, $x^3 - y^3 = (x-y)^3 + 3xy(x-y)$

Put $x = y$ *and* $y = \frac{1}{y}$, $y^3 - \frac{1}{y^3} = \left(y - \frac{1}{y}\right)^3 + 3\left(y - \frac{1}{y}\right)$

$= (9)^3 + 3(9) = 729 + 27 = 756$

The following identity is frequently used in higher studies

$$x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Note If $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$

Example 1.11

Simplify $(x + 2y + 3z)(x^2 + 4y^2 + 9z^2 - 2xy - 6yz - 3zx)$

Solution We know that, $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + y^3 + z^3 - 3xyz$

$\therefore$ $(x + 2y + 3z)(x^2 + 4y^2 + 9z^2 - 2xy - 6yz - 3zx)$

$= (x + 2y + 3z)[x^2 + (2y)^2 + (3z)^2 - (x)(2y) - (2y)(3z) - (3z)(x)]$

$= (x)^3 + (2y)^3 + (3z)^3 - 3(x)(2y)(3z)$

$= x^3 + 8y^3 + 27z^3 - 18xyz$

Example 1.12

Evaluate $12^3 + 13^3 - 25^3$

Solution Let $x = 12,\ y = 13,\ z = -25$. Then

$$x + y + z = 12 + 13 - 25 = 0$$

If $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

$$\therefore\ 12^3 + 13^3 - 25^3 = 12^3 + 13^3 + (-25)^3 = 3(12)(13)(-25) = -11700$$

Exercise 1.1

1. Expand the following

   (i) $(5x + 2y + 3z)^2$ (ii) $(2a + 3b - c)^2$ (iii) $(x - 2y - 4z)^2$ (iv) $(p - 2q + r)^2$

2. Find the expansion of

   (i) $(x + 1)(x + 4)(x + 7)$ (ii) $(p + 2)(p - 4)(p + 6)$

   (iii) $(x + 5)(x - 3)(x - 1)$ (iv) $(x - a)(x - 2a)(x - 4a)$

   (v) $(3x + 1)(3x + 2)(3x + 5)$ (vi) $(2x + 3)(2x - 5)(2x - 7)$

3. Using algebraic identities find the coefficients of x^2 term, x term and constant term.

   (i) $(x + 7)(x + 3)(x + 9)$ (ii) $(x - 5)(x - 4)(x + 2)$

   (iii) $(2x + 3)(2x + 5)(2x + 7)$ (iv) $(5x + 2)(1 - 5x)(5x + 3)$

4. If $(x+a)(x+b)(x+c) \equiv x^3 - 10x^2 + 45x - 15$ find $a+b+c$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $a^2+b^2+c^2$.

5. Expand : (i) $(3a+5b)^3$ (ii) $(4x-3y)^3$ (iii) $\left(2y-\frac{3}{y}\right)^3$

6. Evaluate : (i) 99^3 (ii) 101^3 (iii) 98^3 (iv) 102^3 (v) 1002^3

7. Find $8x^3 + 27y^3$ if $2x + 3y = 13$ and $xy = 6$.

8. If $x - y = -6$ and $xy = 4$, find the value of $x^3 - y^3$.

9. If $x + \frac{1}{x} = 4$, find the value of $x^3 + \frac{1}{x^3}$.

10. If $x - \frac{1}{x} = 3$, find the value of $x^3 - \frac{1}{x^3}$.

11. Simplify : (i) $(2x+y+4z)(4x^2+y^2+16z^2-2xy-4yz-8zx)$

    (ii) $(x-3y-5z)(x^2+9y^2+25z^2+3xy-15yz+5zx)$

12. Evaluate using identities : (i) $6^3 - 9^3 + 3^3$ (ii) $16^3 - 6^3 - 10^3$

1.3 Factorization of Polynomials

We have seen how the distributive property may be used to expand a product of algebraic expressions into sum or difference of expressions.

For example,

(i) $x(x+y) = x^2 + xy$ (ii) $x(y-z) = xy - xz$

(iii) $a(a^2 - 2a + 1) = a^3 - 2a^2 + a$

Now, we will learn how to convert a sum or difference of expressions into a product of expressions.

Now, consider $ab + ac$. Using the distributive law, $a(b+c) = ab + ac$, by writing in the reverse direction $ab + ac$ is $a(b+c)$. This process of expressing $ab + ac$ into $a(b+c)$ is known as factorization. In both the terms, ab and ac 'a' is the common factor. Similarly,

$$5m + 15 = 5(m) + 5(3) = 5(m+3).$$

In $b(b-5) + g(b-5)$ clearly $(b-5)$ is a common factor.

$$b(b-5) + g(b-5) = (b-5)(b+g)$$

Example 1.13

Factorize the following

(i) $pq + pr - 3ps$ (ii) $4a - 8b + 5ax - 10bx$ (iii) $2a^3 + 4a^2$ (iv) $6a^5 - 18a^3 + 42a^2$

Solution

(i) $pq + pr - 3ps = p(q + r - 3s)$

(ii) $4a - 8b + 5ax - 10bx = (4a - 8b) + (5ax - 10bx)$

$= 4(a - 2b) + 5x(a - 2b) = (a - 2b)(4 + 5x)$

(iii) $2a^3 + 4a^2$

Highest common factor is $2a^2$

$\therefore 2a^3 + 4a^2 = 2a^2(a + 2)$.

(iv) $6a^5 - 18a^3 + 42a^2$

Highest common factor is $6a^2$

$\therefore 6a^5 - 18a^3 + 42a^2 = 6a^2(a^3 - 3a + 7)$

1.3.1 Factorization Using Identities

(i) $a^2 + 2ab + b^2 \equiv (a + b)^2$

(ii) $a^2 - 2ab + b^2 \equiv (a - b)^2$ (or) $a^2 - 2ab + b^2 \equiv (-a + b)^2$

(iii) $a^2 - b^2 \equiv (a + b)(a - b)$

(iv) $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \equiv (a + b + c)^2$

Example 1.14

Factorize (i) $4x^2 + 12xy + 9y^2$ (ii) $16a^2 - 8a + 1$ (iii) $9a^2 - 16b^2$

(iv) $(a + b)^2 - (a - b)^2$ (v) $25(a + 2b - 3c)^2 - 9(2a - b - c)^2$ (vi) $x^5 - x$

Solution

(i) $4x^2 + 12xy + 9y^2 = (2x)^2 + 2(2x)(3y) + (3y)^2 = (2x + 3y)^2$

(ii) $16a^2 - 8a + 1 = (4a)^2 - 2(4a)(1) + (1)^2 = (4a - 1)^2$ or $(1 - 4a)^2$

(iii) $9a^2 - 16b^2 = (3a)^2 - (4b)^2 = (3a + 4b)(3a - 4b)$

(iv) $(a+b)^2-(a-b)^2 = [(a+b)+(a-b)][(a+b)-(a-b)]$

$= (a+b+a-b)(a+b-a+b) = (2a)(2b) = (4)(a)(b)$

(v) $25(a+2b-3c)^2-9(2a-b-c)^2 = [5(a+2b-3c)]^2-[3(2a-b-c)]^2$

$= [5(a+2b-3c)+3(2a-b-c)][5(a+2b-3c)-3(2a-b-c)]$

$= (5a+10b-15c+6a-3b-3c)(5a+10b-15c-6a+3b+3c)$

$= (11a+7b-18c)(-a+13b-12c)$

(vi) $x^5-x = x(x^4-1) = x[(x^2)^2-(1)^2]$

$= x(x^2+1)(x^2-1) = x(x^2+1)[(x)^2-(1)^2]$

$= x(x^2+1)(x+1)(x-1)$

1.3.2 Factorization Using the Identity

$a^2+b^2+c^2+2ab+2bc+2ca \equiv (a+b+c)^2$

Example 1.15

Factorize $a^2+4b^2+36-4ab-24b+12a$

Solution $a^2+4b^2+36-4ab-24b+12a$ can be written as

$(a)^2+(-2b)^2+(6)^2+2(a)(-2b)+2(-2b)(6)+2(6)(a) = (a-2b+6)^2$ or

$(-a)^2+(2b)^2+(-6)^2+2(-a)(2b)+2(2b)(-6)+2(-6)(-a) = (-a+2b-6)^2$

That is $(a-2b+6)^2 = [(-1)(-a+2b-6)^2] = (-1)^2(-a+2b-6)^2 = (-a+2b-6)^2$

Example 1.16

Factorize $4x^2+y^2+9z^2-4xy+6yz-12zx$

Solution $4x^2+y^2+9z^2-4xy+6yz-12zx$

$= (2x)^2+(-y)^2+(-3z)^2+2(2x)(-y)+2(-y)(-3z)+2(-3z)(2x)$

$= (2x-y-3z)^2$ or $(-2x+y+3z)^2$

1.3.3 Factorization of $x^3 + y^3$ and $x^3 - y^3$

We have $x^3 + 3x^2y + 3xy^2 + y^3 = (x+y)^3$. So,

$$x^3 + y^3 + 3xy(x+y) = (x+y)^3$$

$$\begin{aligned} \Longrightarrow x^3 + y^3 &= (x+y)^3 - 3xy(x+y) \\ &= (x+y)[(x+y)^2 - 3xy] \\ &= (x+y)(x^2 + 2xy + y^2 - 3xy) \\ &= (x+y)(x^2 - xy + y^2) \end{aligned}$$

$$\boxed{x^3 + y^3 \equiv (x+y)(x^2 - xy + y^2)}$$

We have $x^3 - 3x^2y + 3xy^2 - y^3 = (x-y)^3$. So,

$$x^3 - y^3 - 3xy(x-y) = (x-y)^3$$

$$\begin{aligned} \Longrightarrow x^3 - y^3 &= (x-y)^3 + 3xy(x-y) \\ &= (x-y)[(x-y)^2 + 3xy] \\ &= (x-y)(x^2 - 2xy + y^2 + 3xy) \\ &= (x-y)(x^2 + xy + y^2) \end{aligned}$$

$$\boxed{x^3 - y^3 \equiv (x-y)(x^2 + xy + y^2)}$$

Using the above identities let us factorize the following expressions.

Example 1.17

Factorize (i) $8x^3 + 125y^3$ (ii) $27x^3 - 64y^3$

Solution

(i) $$\begin{aligned} 8x^3 + 125y^3 &= (2x)^3 + (5y)^3 \\ &= (2x+5y)[(2x)^2 - (2x)(5y) + (5y)^2] \\ &= (2x+5y)(4x^2 - 10xy + 25y^2) \end{aligned}$$

(ii) $$\begin{aligned} 27x^3 - 64y^3 &= (3x)^3 - (4y)^3 \\ &= (3x-4y)[(3x)^2 + (3x)(4y) + (4y)^2] \\ &= (3x-4y)(9x^2 + 12xy + 16y^2) \end{aligned}$$

Exercise 1.2

1. Factorize the following expressions:

(i) $2a^3 - 3a^2b + 2a^2c$ (ii) $16x + 64x^2y$ (iii) $10x^3 - 25x^4y$

(iv) $xy - xz + ay - az$ (v) $p^2 + pq + pr + qr$

2. Factorize the following expressions:

(i) $x^2 + 2x + 1$ (ii) $9x^2 - 24xy + 16y^2$

(iii) $b^2 - 4$ (iv) $1 - 36x^2$

3. Factorize the following expressions:

(i) $p^2 + q^2 + r^2 + 2pq + 2qr + 2rp$ (ii) $a^2 + 4b^2 + 36 - 4ab + 24b - 12a$

(iii) $9x^2 + y^2 + 1 - 6xy + 6x - 2y$ (iv) $4a^2 + b^2 + 9c^2 - 4ab - 6bc + 12ca$

(v) $25x^2 + 4y^2 + 9z^2 - 20xy + 12yz - 30zx$

4. Factorize the following expressions:

(i) $27x^3 + 64y^3$ (ii) $m^3 + 8$ (iii) $a^3 + 125$

(iv) $8x^3 - 27y^3$ (v) $x^3 - 8y^3$

1.3.4 Factorization of the Quadratic Polynomials of the type $ax^2 + bx + c$; $a \neq 0$

So far we have used the identities to factorize certain types of polynomials. In this section we will learn, without identities how to resolve quadratic polynomials into two linear polynomials when (i) $a = 1$ and (ii) $a \neq 1$

(i) Factorizing the quadratic polynomials of the type $x^2 + bx + c$.

suppose $(x + p)$ and $(x + q)$ are the two factors of $x^2 + bx + c$. Then we have

$$x^2 + bx + c = (x + p)(x + q)$$
$$= x(x + p) + q(x + p)$$
$$= x^2 + px + qx + pq$$
$$= x^2 + (p + q)x + pq$$

This implies that the two numbers p and q are chosen in such way that $c = pq$ and $b = p + q$ to get $x^2 + bx + c = (x + p)(x + q)$

We use this basic idea to factorize the following problems

For example,

(1) $x^2 + 8x + 15 = (x+3)(x+5)$

here $c = 15 = 3 \times 5$ and $3 + 5 = 8 = b$

(2) $x^2 - 5x + 6 = (x-2)(x-3)$

here $c = 6 = (-2) \times (-3)$ and $(-2) + (-3) = -5 = b$

(3) $x^2 + x - 2 = (x+2)(x-1)$

here $c = -2 = (+2) \times (-1)$ and $(+2) + (-1) = 1 = b$

(4) $x^2 - 4x - 12 = (x-6)(x+2)$

here $c = -12 = (-6) \times (+2)$ and $(-6) + (+2) = -4 = b$

In the above examples the constant term is split into two factors such that their sum is equal to the coefficients of x.

Example 1.18

Factorize the following.

(i) $x^2 + 9x + 14$ (ii) $x^2 - 9x + 14$ (iii) $x^2 + 2x - 15$ (iv) $x^2 - 2x - 15$

Solution

(i) $x^2 + 9x + 14$

To factorize we have to find p and q, such that $pq = 14$ and $p + q = 9$.

$$x^2 + 9x + 14 = x^2 + 2x + 7x + 14$$
$$= x(x+2) + 7(x+2)$$
$$= (x+2)(x+7)$$
$$\therefore x^2 + 9x + 14 = (x+7)(x+2)$$

Factors of 14	Sum of factors
1, 14	15
2, 7	9
The required factors are 2, 7	

(ii) $x^2 - 9x + 14$

To factorize we have to find p and q such that $pq = 14$ and $p + q = -9$

$$x^2 - 9x + 14 = x^2 - 2x - 7x + 14$$
$$= x(x-2) - 7(x-2)$$
$$= (x-2)(x-7)$$
$$\therefore x^2 - 9x + 14 = (x-2)(x-7)$$

Factors of 14	Sum of factors
–1, –14	–15
–2, –7	–9
The required factors are –2, –7	

(iii) $x^2 + 2x - 15$

To factorize we have to find p and q, such that $pq = -15$ and $p + q = 2$

$$x^2 + 2x - 15 = x^2 - 3x + 5x - 15$$
$$= x(x-3) + 5(x-3)$$
$$= (x-3)(x+5)$$
$$\therefore x^2 + 2x - 15 = (x-3)(x+5)$$

Factors of –15	Sum of factors
–1, 15	14
–3, 5	2
The required factors are –3, 5	

(iv) $x^2 - 2x - 15$

To factorize we have to find p and q, such that $pq = -15$ and $p + q = -2$

$$x^2 - 2x - 15 = x^2 + 3x - 5x - 15$$
$$= x(x+3) - 5(x+3)$$
$$= (x+3)(x-5)$$
$$\therefore x^2 - 2x - 15 = (x+3)(x-5)$$

Factors of –15	Sum of factors
1, –15	–14
3, –5	–2
The required factors are 3, –5	

(ii) Factorizing the quadratic polynomials of the type $ax^2 + bx + c$.

Since a is different from 1, the linear factors of $ax^2 + bx + c$ will be of the form $(rx + p)$ and $(sx + q)$.

$$\begin{aligned} \text{Then, } ax^2 + bx + c &= (rx + p)(sx + q) \\ &= rsx^2 + (ps + qr)x + pq \end{aligned}$$

Comparing the coefficients of x^2, we get $a = rs$. Similarly, comparing the coefficients of x, we get $b = ps + qr$. And, on comparing the constant terms, we get $c = pq$.

This shows us that b is the sum of two numbers ps and qr, whose product is $(ps)\times(qr) = (pr)\times(sq) = ac$

Therefore, to factorize $ax^2 + bx + c$, we have to write b as the sum of two numbers whose product is ac.

The following steps to be followed to factorize $ax^2 + bx + c$

Step1 : Multiply the coefficient of x^2 and constant term.

Step2 : Split this product into two factors such that their sum is equal to the coefficient of x.

Step3 : The terms are grouped into two pairs and factorize.

Example 1.19

Factorize the following

(i) $2x^2 + 15x + 27$ (ii) $2x^2 - 15x + 27$

(iii) $2x^2 + 15x - 27$ (iv) $2x^2 - 15x - 27$

Solution

(i) $2x^2 + 15x + 27$

Coefficient of $x^2 = 2$; constant term $= 27$

Their product $= 2 \times 27 = 54$

Coefficient of $x = 15$

$\therefore$ product $= 54$; sum $= 15$

Factors of 54	Sum of factors
1, 54	55
2, 27	29
3, 18	21
6, 9	15
The required factors are 6, 9	

$$\begin{aligned} 2x^2 + 15x + 27 &= 2x^2 + 6x + 9x + 27 \\ &= 2x(x + 3) + 9(x + 3) \\ &= (x + 3)(2x + 9) \end{aligned}$$

$\therefore\ 2x^2 + 15x + 27 = (x + 3)(2x + 9)$

(ii) $2x^2 - 15x + 27$

Coefficient of $x^2 = 2$; constant term $= 27$

Their product$= 2 \times 27 = 54$

Coefficient of $x = -15$

$\therefore$ product$= 54$; sum $= -15$

Factors of 54	Sum of factors
–1, –54	–55
–2, –27	–29
–3, –18	–21
–6, –9	–15
The required factors are –6, –9	

$$2x^2 - 15x + 27 = 2x^2 - 6x - 9x + 27$$
$$= 2x(x-3) - 9(x-3)$$
$$= (x-3)(2x-9)$$
$$\therefore 2x^2 - 15x + 27 = (x-3)(2x-9)$$

(iii) $2x^2 + 15x - 27$

Coefficient of $x^2 = 2$; constant term $= -27$

Their product $= 2 \times -27 = -54$

Coefficient of $x = 15$

$\therefore$ product $= -54$; sum $= 15$

Factors of –54	Sum of factors
–1, 54	53
–2, 27	25
–3, 18	15
The required factors are –3, 18	

$$2x^2 + 15x - 27 = 2x^2 - 3x + 18x - 27$$
$$= x(2x-3) + 9(2x-3)$$
$$= (2x-3)(x+9)$$
$$\therefore 2x^2 + 15x - 27 = (2x-3)(x+9)$$

(iv) $2x^2 - 15x - 27$

Coefficient of $x^2 = 2$; constant term $= -27$

Their product $= 2 \times -27 = -54$

Coefficient of $x = -15$

$\therefore$ product $= -54$; sum $= -15$

Factors of –54	Sum of factors
1, –54	–53
2, –27	–25
3, –18	–15
The required factors are 3, –18	

$$2x^2 - 15x - 27 = 2x^2 + 3x - 18x - 27$$
$$= x(2x+3) - 9(2x+3)$$
$$= (2x+3)(x-9)$$
$$\therefore 2x^2 - 15x - 27 = (2x+3)(x-9)$$

Example 1.20

Factorize $(x+y)^2+9(x+y)+8$

Solution Let $x+y=p$

Then the equation is p^2+9p+8

Coefficient of $p^2=1$; constant term $=8$

Their product$=1\times 8=8$

Coefficient of $p=9$

Factors of 8	Sum of factors
1, 8	9
The required factors are 1, 8	

$\therefore$ product $=8$; sum $=9$

$$p^2+9p+8=p^2+p+8p+8$$
$$=p(p+1)+8(p+1)$$
$$=(p+1)(p+8)$$

substituting, $p=x+y$

$$\therefore\ (x+y)^2+9(x+y)+8=(x+y+1)(x+y+8)$$

Example 1.21

Factorize : (i) x^3-2x^2-x+2 (ii) x^3+3x^2-x-3

Solution

(i) Let $p(x)=x^3-2x^2-x+2$

$p(x)$ is a cubic polynomial, so it may have three linear factors.

The constant term is 2. The factors of 2 are -1, 1, -2 and 2.

$p(-1)=(-1)^3-2(-1)^2-(-1)+2=-1-2+1+2=0$

$\therefore\ (x+1)$ is a factor of $p(x)$.

$p(1)=(1)^3-2(1)^2-1+2=1-2-1+2=0$

$\therefore\ (x-1)$ is a factor of $p(x)$.

$p(-2)=(-2)^3-2(-2)^2-(-2)+2=-8-8+2+2=-12\neq 0$

$\therefore\ (x+2)$ is not a factor of $p(x)$.

$p(2)=(2)^3-2(2)^2-2+2=8-8-2+2=0$

$\therefore\ (x-2)$ is a factor of $p(x)$.

The three factors of $p(x)$ are $(x+1),(x-1)$ and $(x-2)$

$\therefore\ x^3-2x^2-x+2=(x+1)(x-1)(x-2)$.

Another method

$$x^3-2x^2-x+2=x^2(x-2)-1(x-2)$$

$= (x-2)(x^2-1)$

$= (x-2)(x+1)(x-1)$ $\quad [(\because a^2 - b^2 = (a+b)(a-b)]$

(ii) Let $p(x) = x^3 + 3x^2 - x - 3$

$p(x)$ is a cubic polynomial, so it may have three linear factors.

The constant term is –3. The factors of –3 are –1, 1,–3 and 3.

$p(-1) = (-1)^3 + 3(-1)^2 - (-1) - 3 = -1 + 3 + 1 - 3 = 0$

$\therefore (x+1)$ is a factor of $p(x)$.

$p(1) = (1)^3 + 3(1)^2 - 1 - 3 = 1 + 3 - 1 - 3 = 0$

$\therefore (x-1)$ is a factor of $p(x)$.

$p(-3) = (-3)^3 + 3(-3)^2 - (-3) - 3 = -27 + 27 + 3 - 3 = 0$

$\therefore (x+3)$ is a factor of $p(x)$.

The three factors of $p(x)$ are $(x+1), (x-1)$ and $(x+3)$

$\therefore x^3 + 3x^2 - x - 3 = (x+1)(x-1)(x+3)$.

Exercise 1.3

1. Factorize each of the following.

(i) $x^2 + 15x + 14$ (ii) $x^2 + 13x + 30$ (iii) $y^2 + 7y + 12$

(iv) $x^2 - 14x + 24$ (v) $y^2 - 16y + 60$ (vi) $t^2 - 17t + 72$

(vii) $x^2 + 14x - 15$ (viii) $x^2 + 9x - 22$ (ix) $y^2 + 5y - 36$

(x) $x^2 - 2x - 99$ (xi) $m^2 - 10m - 144$ (xii) $y^2 - y - 20$

2. Factorize each of the following.

(i) $3x^2 + 19x + 6$ (ii) $5x^2 + 22x + 8$ (iii) $2x^2 + 9x + 10$

(iv) $14x^2 + 31x + 6$ (v) $5y^2 - 29y + 20$ (vi) $9y^2 - 16y + 7$

(vii) $6x^2 - 5x + 1$ (viii) $3x^2 - 10x + 8$ (ix) $3x^2 + 5x - 2$

(x) $2a^2 + 17a - 30$ (xi) $11 + 5x - 6x^2$ (xii) $8x^2 + 29x - 12$

(xiii) $2x^2 - 3x - 14$ (xiv) $18x^2 - x - 4$ (xv) $10 - 7x - 3x^2$

3. Factorize the following

(i) $(a+b)^2 + 9(a+b) + 14$ (ii) $(p-q)^2 - 7(p-q) - 18$

4. Factorize the following

(i) $x^3 + 2x^2 - x - 2$ (ii) $x^3 - 3x^2 - x + 3$

(iii) $x^3 + x^2 - 4x - 4$ (iv) $x^3 + 5x^2 - x - 5$

Example 1.22

Solve the following pair of equations by substitution method.

$$2x + 5y = 2 \text{ and } x + 2y = 3$$

Solution We have $2x + 5y = 2$ (1)

$x + 2y = 3$ (2)

Equation (2) becomes, $x = 3 - 2y$ (3)

Substituting x in (1) we get, $2(3 - 2y) + 5y = 2$

$$\Longrightarrow 6 - 4y + 5y = 2$$

$$-4y + 5y = 2 - 6$$

$$\therefore y = -4$$

Substituting $y = -4$ in (3), we get, $x = 3 - 2(-4) = 3 + 8 = 11$

$\therefore$ The solution is $x = 11$ and $y = -4$

Example 1.23

Solve $x + 3y = 16$, $2x - y = 4$ by using substitution method.

Solution

We have $x + 3y = 16$ (1)

$2x - y = 4$ (2)

Equation (1) becomes, $x = 16 - 3y$ (3)

Substituting x in (2) we get, $2(16 - 3y) - y = 4$

$$\Longrightarrow 32 - 6y - y = 4$$

$$-6y - y = 4 - 32$$

$$-7y = -28$$

$$y = \frac{-28}{-7} = 4$$

Substituting $y = 4$ in (3) we get, $x = 16 - 3(4)$

$$= 16 - 12 = 4$$

$\therefore$ The solution is $x = 4$ and $y = 4$.

Example 1.24

Solve by substitution method $\frac{1}{x}+\frac{1}{y}=4$ and $\frac{2}{x}+\frac{3}{y}=7, \quad x \neq 0, y \neq 0$

Solution

Let $\frac{1}{x}=a$ and $\frac{1}{y}=b$

The given equations become

$$a+b=4 \qquad (1)$$

$$2a+3b=7 \qquad (2)$$

Equation (1) becomes $b=4-a$ (3)

Substituting b in (2) we get, $2a+3(4-a)=7$

$$\Longrightarrow 2a+12-3a=7$$

$$2a-3a=7-12$$

$$-a=-5 \Longrightarrow a=5$$

Substituting $a=5$ in (3) we get, $b=4-5=-1$

But $\frac{1}{x}=a \Rightarrow x=\frac{1}{a}=\frac{1}{5}$

$$\frac{1}{y}=b \Rightarrow y=\frac{1}{b}=\frac{1}{-1}=-1$$

$\therefore$ The solution is $x=\frac{1}{5},\ y=-1$

Example 1.25

The cost of a pen and a note book is ₹ 60. The cost of a pen is ₹ 10 less than that of a notebook. Find the cost of each.

Solution

Let the cost of a pen = ₹ x

Let the cost of a note book = ₹ y

From given data we have

$x + y = 60 \quad (1)$

$x = y - 10 \quad (2)$

Substituting x in (1) we get, $y - 10 + y = 60$

$\Longrightarrow y + y = 60 + 10 \Longrightarrow 2y = 70$

$\therefore y = \frac{70}{2} = 35$

Substituting $y = 35$ in (2) we get, $x = 35 - 10 = 25$

$\therefore$ The cost of a pen is ₹ 25.

The cost of a note book is ₹ 35.

Example 1.26

The cost of three mathematics books and four science books is ₹ 216. The cost of three mathematics books is the same as that of four science books. Find the cost of each book.

Solution

Let the cost of a mathematics book be ₹ x and cost of a science book be ₹ y.

By given data,

$3x + 4y = 216 \quad (1)$

$3x = 4y \quad (2)$

The equation (2) becomes, $x = \frac{4y}{3} \quad (3)$

Substituting x in (1) we get, $3\left(\frac{4y}{3}\right) + 4y = 216$

$\Longrightarrow 4y + 4y = 216 \Longrightarrow 8y = 216$

$$\therefore y = \frac{216}{8} = 27$$

substituting $y = 27$ in (3) we get, $x = \frac{4(27)}{3} = 36$

$\therefore$ The cost of one mathematics book = ₹ 36.

The cost of one science book = ₹ 27.

Example 1.27

From Dharmapuri bus stand if we buy 2 tickets to Palacode and 3 tickets to Karimangalam the total cost is ₹ 32, but if we buy 3 tickets to Palacode and one ticket to Karimangalam the total cost is ₹ 27. Find the fares from Dharmapuri to Palacode and to Karimangalam.

Solution

Let the fare from Dharmapuri to Palacode be ₹ x and to Karimangalam be ₹ y.

From the given data, we have

$2x + 3y = 32$ (1)

$3x + y = 27$ (2)

Equation (2) becomes, $y = 27 - 3x$ (3)

Substituting y in (1) we get, $2x + 3(27 - 3x) = 32$

$$\Longrightarrow 2x + 81 - 9x = 32$$

$$2x - 9x = 32 - 81$$

$$-7x = -49$$

$$-7x = -49$$

$$\therefore x = \frac{-49}{-7} = 7$$

Substituting $x = 7$ in (3) we get, $y = 27 - 3(7) = 27 - 21 = 6$

$\therefore$ The fare from Dharmapuri to Palacode is ₹ 7 and to Karimangalam is ₹ 6.

Example 1.28

The sum of two numbers is 55 and their difference is 7. Find the numbers .

Solution

Let the two numbers be x and y, where $x > y$

By the given data, $x + y = 55$ (1)

$x - y = 7$ (2)

Equation (2) becomes, $x = 7 + y$ (3)

Substituting x in (1) we get, $7 + y + y = 55$

$$\Longrightarrow 2y = 55 - 7 = 48$$

$$\therefore y = \frac{48}{2} = 24$$

Substituting $y = 24$ in (3) we get, $x = 7 + 24 = 31$.

$\therefore$ The required two numbers are 31 and 24.

Example 1.29

A number consist of two digits whose sum is 11. The number formed by reversing the digits is 9 less than the original number. Find the number.

Solution

Let the tens digit be x and the units digit be y. Then the number is $10x + y$.

Sum of the digits is $x + y = 11$ (1)

The number formed by reversing the digits is $10y + x$.

Given data, $(10x + y) - 9 = 10y + x$

$$\Longrightarrow 10x + y - 10y - x = 9$$

$$9x - 9y = 9$$

Dividing by 9 on both sides, $x - y = 1$ (2)

Equation (2) becomes $x = 1 + y$ (3)

Substituting x in (1) we get, $1 + y + y = 11$

$$\Longrightarrow 2y + 1 = 11$$

$$2y = 11 - 1 = 10$$

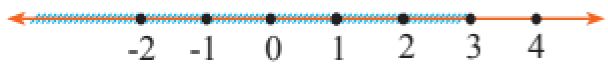
$$\therefore y = \frac{10}{2} = 5$$

Substituting $y = 5$ in (3) we get, $x = 1 + 5 = 6$

$\therefore$ The number is $10x + y = 10(6) + 5 = 65$

Example 1.30

Solve $4(x-1) \le 8$



Solution

$4(x-1) \le 8$

Dividing by 4 on both sides,

$x - 1 \le 2$

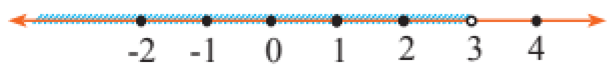
$\Longrightarrow \quad x \le 2 + 1 \Longrightarrow x \le 3$

The real numbers less than or equal to 3 are solutions of given inequation.

Shaded circle indicates that point is included in the solution set.

Example 1.31

Solve $3(5-x) > 6$



Solution We have, $3(5-x) > 6$

Dividing by 3 on both sides, $5 - x > 2$

$\Longrightarrow -x > 2 - 5 \Longrightarrow -x > -3$

$\therefore \; x < 3$ (See remark given below)

The real numbers less than 3 are solutions of given inequation.

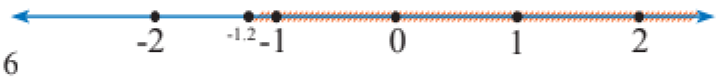
Remark (i) $-a > -b \Longrightarrow a < b$ (ii) $a < b \Longrightarrow \frac{1}{a} > \frac{1}{b}$ where $a \neq 0,\ b \neq 0$

(iii) $a < b \Longrightarrow ka < kb$ for $k > 0$ (iv) $a < b \Longrightarrow ka > kb$ for $k < 0$

Example 1.32

Solve $3 - 5x \leq 9$

Solution We have, $3 - 5x \leq 9$



$\Longrightarrow -5x \leq 9 - 3 \Longrightarrow -5x \leq 6$

$\Longrightarrow 5x \geq -6 \Longrightarrow x \geq -\frac{6}{5} \Longrightarrow x \geq -1.2$

The real numbers greater than or equal to -1.2 are solutions of given inequation.

Exercise 1.4

1. Solve the following equations by substitution method .

   (i) $x + 3y = 10;\ 2x + y = 5$ (ii) $2x + y = 1;\ 3x - 4y = 18$

   (iii) $5x + 3y = 21;\ 2x - y = 4$ (iv) $\frac{1}{x} + \frac{2}{y} = 9;\ \frac{2}{x} + \frac{1}{y} = 12 \quad (x \neq 0, y \neq 0)$

   (v) $\frac{3}{x} + \frac{1}{y} = 7;\ \frac{5}{x} - \frac{4}{y} = 6 \ (x \neq 0, y \neq 0)$

2. Find two numbers whose sum is 24 and difference is 8.

3. A number consists of two digits whose sum is 9. The number formed by reversing the digits exceeds twice the original number by 18. Find the original number.

4 Kavi and Kural each had a number of apples . Kavi said to Kural "If you give me 4 of your apples, my number will be thrice yours". Kural replied "If you give me 26, my number will be twice yours". How many did each have with them?.

5. Solve the following inequations.

   (i) $2x + 7 > 15$ (ii) $2(x - 2) < 3$ (iii) $2(x + 7) \leq 9$ (iv) $3x + 14 \geq 8$

- $(x + y + z)^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$
- $(x + y)^3 \equiv x^3 + y^3 + 3xy(x + y)$ $\quad x^3 + y^3 \equiv (x + y)(x^2 - xy + y^2)$
- $(x - y)^3 \equiv x^3 - y^3 - 3xy(x - y)$ $\quad x^3 - y^3 \equiv (x - y)(x^2 + xy + y^2)$
- $x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
- $(x + a)(x + b)(x + c) \equiv x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$

2.2.2 Pythagoras Theorem

The Pythagoras theorem is a tool to solve for unknown values on right triangle.

Pythagoras Theorem: The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

This relationship is useful in solving many problems and in developing trigonometric concepts.

2.2.3 Trigonometric Ratios

Consider the right triangle in the Fig. 2.2. In the right triangle, we refer to the lengths of the three sides according to how they are placed in relation to the angle θ

- The side that is opposite to the right angle is called the *Hypotenuse*. This is the longest side in a right triangle.
- The side that is opposite to the angle θ is called the *Opposite side*.
- The side that runs alongside the angle θ and which is not the Hypotenuse is called the *Adjacent side*.

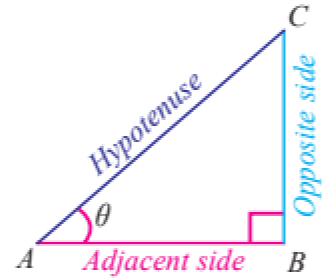


Fig. 2.2

3.3 Mean

3.3.1 Arithmetic Mean - Raw Data

The arithmetic mean is the sum of a set of observations, positive, negative or zero, divided by the number of observations. If we have n real numbers $x_1, x_2, x_3 ...., x_n$, then their arithmetic mean, denoted by $\overline{x}$, is given by

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} \text{ or } \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ or } \overline{x} = \frac{\sum x}{n}$$

Remark $\overline{x} = \frac{\sum x}{n} \implies n\overline{x} = \sum x$. That is,

Total number of observations $\times$ Mean = Sum of all observations

Example 3.4

Find the arithmetic mean of the marks 72, 73, 75, 82, 74 obtained by a student in 5 subjects in an annual examination.

Solution Here n = 5

$$\overline{x} = \frac{\sum x}{n} = \frac{72 + 73 + 75 + 82 + 74}{5} = \frac{376}{5} = 75.2$$

$\therefore$ Mean = 75.2

Example 3.5

The mean of the 5 numbers is 32. If one of the numbers is excluded, then the mean is reduced by 4. Find the excluded number.

Solution

Mean of 5 numbers $= 32$.

Sum of these numbers $= 32 \times 5 = 160$ $\quad (\because n\overline{x} = \sum x)$

Mean of 4 numbers $= 32 - 4 = 28$

Sum of these 4 numbers $= 28 \times 4 = 112$

Excluded number $=$ (Sum of the 5 given numbers) $-$ (Sum of the 4 numbers)

$= 160 - 112 = 48$

Example 3.6

Obtain the mean of the following data.

x	5	10	15	20	25
f	3	10	25	7	5

Solution

x	f	fx
5	3	15
10	10	100
15	25	375
20	7	140
25	5	125
Total	$\sum f = 50$	$\sum fx = 755$

$$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{755}{50} = 15.1$$

Mean $= 15.1$

3.4 Median

Median is defined as the middle item of the given observations arranged in order.

3.4.1 Median - Raw Data

Steps:

(i) Arrange the n given numbers in ascending or descending order of magnitude.

(ii) When n is odd, $\left(\frac{n+1}{2}\right)^{th}$ observation is the median.

(iii) When n is even the median is the arithmetic mean of the two middle values.

i.e., when n is even,

Median = Mean of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations.

Example 3.10

Find the median of the following numbers

(i) 24, 22, 23, 14, 15, 7, 21 (ii) 17, 15, 9, 13, 21, 32, 42, 7, 12, 10.

Solution

(i) Let us arrange the numbers in ascending order as below.

7, 14, 15, 21, 22, 23, 24

Number of items $n = 7$

$$\text{Median} = \left(\frac{n+1}{2}\right)^{th} \text{ observation} \quad (\because n \text{ is odd})$$

$$= \left(\frac{7+1}{2}\right)^{th} \text{ observation}$$

$$= 4^{th} \text{ observation} \quad = 21$$

(ii) Let us arrange the numbers in ascending order

7, 9, 10, 12, 13, 15, 17, 21, 32, 42.

Number of items $n = 10$

Median is the mean of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations. $(\because n$ is even$)$

$$\left(\frac{n}{2}\right)^{th} \text{ observation} = \left(\frac{10}{2}\right)^{th} \text{ observation} = 5^{th} \text{ observation} = 13$$

$$\left(\frac{n}{2}+1\right)^{th} \text{ observation} = 6^{th} \text{ observation} = 15.$$

$$\therefore \text{Median} = \frac{13+15}{2} = 14$$

Exercise 3.5

Choose the Correct Answer

1. The mean of the first 10 natural numbers is
   (A) 25 (B) 55 (C) 5.5 (D) 2.5
2. The Arithmetic mean of integers from –5 to 5 is
   (A) 3 (B) 0 (C) 25 (D) 10
3. If the mean of x, $x+2$, $x+4$, $x+6$, $x+8$ is 20 then x is
   (A) 32 (B) 16 (C) 8 (D) 4
4. The mode of the data 5, 5, 5, 5, 5, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4 is
   (A) 2 (B) 3 (C) 4 (D) 5
5. The median of 14, 12, 10, 9, 11 is
   (A) 11 (B) 10 (C) 9.5 (D) 10.5
6. The median of 2, 7, 4, 8, 9, 1 is
   (A) 4 (B) 6 (C) 5.5 (D) 7
7. The mean of first 5 whole number is
   (A) 2 (B) 2.5 (C) 3 (D) 0
8. The Arithmetic mean of 10 number is –7. If 5 is added to every number, then the new Arithmetic mean is
   (A) –2 (B) 12 (C) –7 (D) 17
9. The Arithmetic mean of all the factors of 24 is
   (A) 8.5 (B) 5.67 (C) 7 (D) 7.5
10. The mean of 5 numbers is 20. If one number is excluded their mean is 15. Then the excluded number is
    (A) 5 (B) 40 (C) 20 (D) 10.

1.2 Surds

We know that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ are irrational numbers. These are square roots of rational numbers, which cannot be expressed as squares of any rational number. $\sqrt[3]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{7}$ etc. are the cube roots of rational numbers, which cannot be expressed as cubes of any rational number. This type of irrational numbers are called surds or radicals.

Key Concept **Surds**

If 'a' is a positive rational number and n is a positive integer such that $\sqrt[n]{a}$ is an irrational number, then $\sqrt[n]{a}$ is called a 'surd' or a 'radical'.

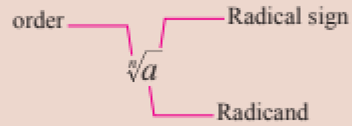
Notation

The general form of a surd is $\sqrt[n]{a}$

$\sqrt{\ }$ is called the *radical sign*

n is called the *order* of the radical.

a is called the *radicand*.



1.2.1 Index Form of a Surd

The index form of a surd $\sqrt[n]{a}$ is $a^{\frac{1}{n}}$

Think and Answer !
$a^{\frac{1}{3}}$ and a^3 differ. why?

For example, $\sqrt[5]{8}$ can be written in index form as $\sqrt[5]{8} = (8)^{\frac{1}{5}}$

In the following table, the index form, order and radicand of some surds are given.

Surd	Index Form	Order	Radicand
$\sqrt{5}$	$5^{\frac{1}{2}}$	2	5
$\sqrt[3]{14}$	$(14)^{\frac{1}{3}}$	3	14
$\sqrt[4]{7}$	$7^{\frac{1}{4}}$	4	7
$\sqrt{50}$	$(50)^{\frac{1}{2}}$	2	50
$\sqrt[5]{11}$	$(11)^{\frac{1}{5}}$	5	11

Remark If $\sqrt[n]{a}$ is a surd, then

(i) a is a positive rational number. (ii) $\sqrt[n]{a}$ is an irrational number.

In the table given below both the columns A and B have irrational numbers.

A	B
$\sqrt{5}$	$\sqrt{2+\sqrt{3}}$
$\sqrt[3]{7}$	$\sqrt[3]{5+\sqrt{7}}$
$\sqrt[3]{100}$	$\sqrt[3]{10-\sqrt[3]{3}}$
$\sqrt{12}$	$\sqrt[4]{15+\sqrt{5}}$

The numbers in Column A are surds and the numbers in Column B are irrationals.

Thus, every surd is an irrational number, but every irrational number need not be a surd.

1.2.2 Reduction of a Surd to its Simplest Form

We can reduce a surd to its simplest form.

For example, consider the surd $\sqrt{50}$

$$\text{Now } \sqrt{50} = \sqrt{25\times 2} = \sqrt{25}\sqrt{2} = \sqrt{5^2}\sqrt{2} = 5\sqrt{2}$$

Thus $5\sqrt{2}$ is the simplest form of $\sqrt{50}$.

1.2.3 Like and Unlike Surds

Surds in their simplest form are called like surds if their order and radicand are the same. Otherwise the surds are called unlike surds.

For example,

(i) $\sqrt{5}, 4\sqrt{5}, -6\sqrt{5}$ are like surds. (ii) $\sqrt{10}, \sqrt[3]{3}, \sqrt[4]{5}, \sqrt[3]{81}$ are unlike surds.

1.2.4 Pure surds

A Surd is called a pure surd if its rational coefficient is unity

For example, $\sqrt{3}, \sqrt[3]{5}, \sqrt[4]{12}, \sqrt{80}$ are pure surds.

1.2.5 Mixed Surds

A Surd is called a mixed if its rational coefficient is other than unity

For example, $2\sqrt{3}, 5\sqrt[3]{5}, 3\sqrt[4]{12}$ are mixed surds.

A mixed surd can be converted into a pure surd and a pure surd may or may not be converted into a mixed surd.

For example,

(i) $\sqrt{80} = \sqrt{16\times 5} = 4\sqrt{5}$ (ii) $3\sqrt{2} = \sqrt{3^2\times 2} = \sqrt{9\times 2} = \sqrt{18}$

(iii) $\sqrt{17}$ is a pure surd, but it cannot be converted into a mixed surd.

Laws of Radicals

For positive integers m, n and positive rational numbers a, b we have

(i) $(\sqrt[n]{a})^n = a = \sqrt[n]{a^n}$ (ii) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

(iii) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$ (iv) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Using (i) we have $(\sqrt{a})^2 = a$, $\sqrt[3]{a^3} = (\sqrt[3]{a})^3 = a$

Example 1.1

Convert the following surds into index form.

(i) $\sqrt{7}$ (ii) $\sqrt[4]{8}$ (iii) $\sqrt[3]{6}$ (iv) $\sqrt[8]{12}$

Solution In index form we write the given surds as follows

(i) $\sqrt{7} = 7^{\frac{1}{2}}$ (ii) $\sqrt[4]{8} = 8^{\frac{1}{4}}$ (iii) $\sqrt[3]{6} = 6^{\frac{1}{3}}$ (iv) $\sqrt[8]{12} = (12)^{\frac{1}{8}}$

Example 1.2

Express the following surds in its simplest form.

(i) $\sqrt[3]{32}$ (ii) $\sqrt{63}$ (iii) $\sqrt{243}$ (iv) $\sqrt[3]{256}$

Solution

(i) $\sqrt[3]{32} = \sqrt[3]{8 \times 4} = \sqrt[3]{8} \times \sqrt[3]{4} = \sqrt[3]{2^3} \times \sqrt[3]{4} = 2\sqrt[3]{4}$

(ii) $\sqrt{63} = \sqrt{9 \times 7} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}$

(iii) $\sqrt{243} = \sqrt{81 \times 3} = \sqrt{81} \times \sqrt{3} = \sqrt{9^2} \times \sqrt{3} = 9\sqrt{3}$

(iv) $\sqrt[3]{256} = \sqrt[3]{64 \times 4} = \sqrt[3]{64} \times \sqrt[3]{4} = \sqrt[3]{4^3} \times \sqrt[3]{4} = 4\sqrt[3]{4}$

Example 1.3

Express the following mixed surds into pure surds.

(i) $16\sqrt{2}$ (ii) $3\sqrt[3]{2}$ (iii) $2\sqrt[4]{5}$ (iv) $6\sqrt{3}$

Solution

(i) $16\sqrt{2} = \sqrt{16^2} \times \sqrt{2}$ $(\because 16 = \sqrt{16^2})$

$= \sqrt{16^2 \times 2} = \sqrt{256 \times 2} = \sqrt{512}$

(ii) $3\sqrt[3]{2} = \sqrt[3]{3^3 \times 2}$ $(\because 3 = \sqrt[3]{3^3})$

$= \sqrt[3]{27 \times 2} = \sqrt[3]{54}$

(iii) $2\sqrt[4]{5} = \sqrt[4]{2^4 \times 5}$ $(\because 2 = \sqrt[4]{2^4})$

$= \sqrt[4]{16 \times 5} = \sqrt[4]{80}$

(iv) $6\sqrt{3} = \sqrt{6^2 \times 3}$ $(\because 6 = \sqrt{6^2})$

$= \sqrt{36 \times 3} = \sqrt{108}$

Example 1.4

Identify whether $\sqrt{32}$ is rational or irrational.

Solution $\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$

4 is a rational number and $\sqrt{2}$ is an irrational number.

$\therefore 4\sqrt{2}$ is an irrational number and hence $\sqrt{32}$ is an irrational number.

Example 1.5

Identify whether the following numbers are rational or irrational.

(i) $3 + \sqrt{3}$ (ii) $(4 + \sqrt{2}) - (4 - \sqrt{3})$ (iii) $\frac{\sqrt{18}}{2\sqrt{2}}$ (iv) $\sqrt{19} - (2 + \sqrt{19})$

(v) $\frac{2}{\sqrt{3}}$ (vi) $\sqrt{12} \times \sqrt{3}$

Solution

(i) $3 + \sqrt{3}$

3 is a rational number and $\sqrt{3}$ is irrational. Hence, $3 + \sqrt{3}$ is irrational.

(ii) $(4 + \sqrt{2}) - (4 - \sqrt{3})$

$= 4 + \sqrt{2} - 4 + \sqrt{3} = \sqrt{2} + \sqrt{3}$, is irrational.

(iii) $\frac{\sqrt{18}}{2\sqrt{2}} = \frac{\sqrt{9 \times 2}}{2\sqrt{2}} = \frac{\sqrt{9} \times \sqrt{2}}{2\sqrt{2}} = \frac{3}{2}$, is rational.

(iv) $\sqrt{19} - (2 + \sqrt{19}) = \sqrt{19} - 2 - \sqrt{19} = -2$, is rational.

(v) $\frac{2}{\sqrt{3}}$ here 2 is rational and $\sqrt{3}$ is irrational. Hence, $\frac{2}{\sqrt{3}}$ is irrational.

(vi) $\sqrt{12} \times \sqrt{3} = \sqrt{12 \times 3} = \sqrt{36} = 6$, is rational.

1.3 Four Basic Operations on Surds

1.3.1 Addition and Subtraction of Surds

Like surds can be added and subtracted.

Example 1.6

Simplify

(i) $10\sqrt{2} - 2\sqrt{2} + 4\sqrt{32}$

(ii) $\sqrt{48} - 3\sqrt{72} - \sqrt{27} + 5\sqrt{18}$

(iii) $\sqrt[3]{16} + 8\sqrt[3]{54} - \sqrt[3]{128}$

Solution

(i) $10\sqrt{2} - 2\sqrt{2} + 4\sqrt{32}$

$= 10\sqrt{2} - 2\sqrt{2} + 4\sqrt{16\times 2}$

$= 10\sqrt{2} - 2\sqrt{2} + 4\times 4\times\sqrt{2}$

$= (10 - 2 + 16)\sqrt{2} = 24\sqrt{2}$

(ii) $\sqrt{48} - 3\sqrt{72} - \sqrt{27} + 5\sqrt{18}$

$= \sqrt{16\times 3} - 3\sqrt{36\times 2} - \sqrt{9\times 3} + 5\sqrt{9\times 2}$

$= \sqrt{16}\sqrt{3} - 3\sqrt{36}\sqrt{2} - \sqrt{9}\sqrt{3} + 5\sqrt{9}\sqrt{2}$

$= 4\sqrt{3} - 18\sqrt{2} - 3\sqrt{3} + 15\sqrt{2}$

$= (-18 + 15)\sqrt{2} + (4 - 3)\sqrt{3} = -3\sqrt{2} + \sqrt{3}$

(iii) $\sqrt[3]{16} + 8\sqrt[3]{54} - \sqrt[3]{128}$

$= \sqrt[3]{8\times 2} + 8\sqrt[3]{27\times 2} - \sqrt[3]{64\times 2}$

$= \sqrt[3]{8}\sqrt[3]{2} + 8\sqrt[3]{27}\sqrt[3]{2} - \sqrt[3]{64}\sqrt[3]{2}$

$= 2\sqrt[3]{2} + 8\times 3\times\sqrt[3]{2} - 4\sqrt[3]{2}$

$= 2\sqrt[3]{2} + 24\sqrt[3]{2} - 4\sqrt[3]{2}$

$= (2 + 24 - 4)\sqrt[3]{2} = 22\sqrt[3]{2}$

1.3.2 Multiplication of Surds

Product of two like surds can simplified using the following law.

$\sqrt[n]{a}\times\sqrt[n]{b} = \sqrt[n]{ab}$

Example 1.7

Multiply (i) $\sqrt[3]{13}\times\sqrt[3]{5}$ (ii) $\sqrt[4]{32}\times\sqrt[4]{8}$

Solution

(i) $\sqrt[3]{13}\times\sqrt[3]{5} = \sqrt[3]{13\times 5} = \sqrt[3]{65}$

(ii) $\sqrt[4]{32}\times\sqrt[4]{8} = \sqrt[4]{32\times 8}$

$= \sqrt[4]{2^5\times 2^3} = \sqrt[4]{2^8} = \sqrt[4]{2^4\times 2^4} = 2\times 2 = 4$

1.3.3 Division of Surds

Like surds can be divided using the law

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example 1.8

Simplify (i) $15\sqrt{54} \div 3\sqrt{6}$ (ii) $\sqrt[3]{128} \div \sqrt[3]{64}$

Solution

(i) $15\sqrt{54} \div 3\sqrt{6}$

$$= \frac{15\sqrt{54}}{3\sqrt{6}} = 5\sqrt{\frac{54}{6}} = 5\sqrt{9} = 5 \times 3 = 15$$

(ii) $\sqrt[3]{128} \div \sqrt[3]{64}$

$$= \frac{\sqrt[3]{128}}{\sqrt[3]{64}} = \sqrt[3]{\frac{128}{64}} = \sqrt[3]{2}$$

Note When the order of the surds are different, we convert them to the same order and then multiplication or division is carried out.

Result $\sqrt[n]{a} = \sqrt[m]{a^{\frac{m}{n}}}$

For example, (i) $\sqrt[3]{5} = \sqrt[12]{5^{\frac{12}{3}}} = \sqrt[12]{5^4}$ (ii) $\sqrt[4]{11} = \sqrt[8]{11^{\frac{8}{4}}} = \sqrt[8]{11^2}$

1.3.4 Comparison of Surds

Irrational numbers of the same order can be compared. Among the irrational numbers of same order, the greatest irrational number is the one with the largest radicand.

If the order of the irrational numbers are not the same, we first convert them to the same order. Then, we just compare the radicands.

Example 1.9

Convert the irrational numbers $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$ to the same order.

Solution The orders of the given irrational numbers are 2, 3 and 4.

LCM of 2, 3 and 4 is 12

$\sqrt{3} = \sqrt[12]{3^6} = \sqrt[12]{729}$

$\sqrt[3]{4} = \sqrt[12]{4^4} = \sqrt[12]{256}$

$\sqrt[4]{5} = \sqrt[12]{5^3} = \sqrt[12]{125}$

Example 1.10

Which is greater ? $\sqrt[4]{5}$ or $\sqrt[3]{4}$

Solution The orders of the given irrational numbers are 3 and 4.

We have to convert each of the irrational number to an irrational number of the same order.

LCM of 3 and 4 is 12. Now we convert each irrational number as of order 12.

$$\sqrt[4]{5} = \sqrt[12]{5^3} = \sqrt[12]{125}$$

$$\sqrt[3]{4} = \sqrt[12]{4^4} = \sqrt[12]{256}$$

$$\therefore \sqrt[12]{256} > \sqrt[12]{125} \quad \Rightarrow \sqrt[3]{4} > \sqrt[4]{5}$$

Example 1.11

Write the irrational numbers $\sqrt[3]{2}$, $\sqrt[4]{4}$, $\sqrt{3}$ in

(i) ascending order (ii) descending order

Solution The orders of the irrational numbers $\sqrt[3]{2}, \sqrt[4]{4}$ and $\sqrt{3}$ are 3, 4 and 2 respectively LCM of 2, 3, and 4 is 12. Now, we convert each irrational number as of order 12.

$$\sqrt[3]{2} = \sqrt[12]{2^4} = \sqrt[12]{16}$$

$$\sqrt[4]{4} = \sqrt[12]{4^3} = \sqrt[12]{64}$$

$$\sqrt{3} = \sqrt[12]{3^6} = \sqrt[12]{729}$$

$\therefore$ Ascending order: $\sqrt[3]{2}, \sqrt[4]{4}, \sqrt{3}$

Descending order: $\sqrt{3}, \sqrt[4]{4}, \sqrt[3]{2}$.

Exercise 1.1

1. Identify which of the following are surds and which are not with reasons.

   (i) $\sqrt{8}\times\sqrt{6}$ (ii) $\sqrt{90}$ (iii) $\sqrt{180}\times\sqrt{5}$ (iv) $4\sqrt{5}\div\sqrt{8}$ (v) $\sqrt[3]{4}\times\sqrt[3]{16}$

2. Simplify

   (i) $(10+\sqrt{3})(2+\sqrt{5})$ (ii) $(\sqrt{5}+\sqrt{3})^2$

   (iii) $(\sqrt{13}-\sqrt{2})(\sqrt{13}+\sqrt{2})$ (iv) $(8+\sqrt{3})(8-\sqrt{3})$

3. Simplify the following.

   (i) $5\sqrt{75}+8\sqrt{108}-\frac{1}{2}\sqrt{48}$ (ii) $7\sqrt[3]{2}+6\sqrt[3]{16}-\sqrt[3]{54}$

   (iii) $4\sqrt{72}-\sqrt{50}-7\sqrt{128}$ (iv) $2\sqrt[3]{40}+3\sqrt[3]{625}-4\sqrt[3]{320}$

4. Express the following surds in its simplest form.

   (i) $\sqrt[3]{108}$ (ii) $\sqrt{98}$ (iii) $\sqrt{192}$ (iv) $\sqrt[3]{625}$

5. Express the following as pure surds.

   (i) $6\sqrt{5}$ (ii) $5\sqrt[3]{4}$ (iii) $3\sqrt[4]{5}$ (iv) $\frac{3}{4}\sqrt{8}$

6. Simplify the following.

   (i) $\sqrt{5}\times\sqrt{18}$ (ii) $\sqrt[3]{7}\times\sqrt[3]{8}$ (iii) $\sqrt[4]{8}\times\sqrt[4]{12}$ (iv) $\sqrt[3]{3}\times\sqrt[6]{5}$

   (v) $3\sqrt{35}\div2\sqrt{7}$ (vi) $\sqrt[4]{48}\div\sqrt[8]{72}$

7. Which is greater ?

   (i) $\sqrt{2}$ or $\sqrt[3]{3}$ (ii) $\sqrt[3]{3}$ or $\sqrt[4]{4}$ (iii) $\sqrt{3}$ or $\sqrt[4]{10}$

8. Arrange in descending and ascending order.

   (i) $\sqrt[4]{5},\sqrt{3},\sqrt[3]{4}$ (ii) $\sqrt[3]{2},\sqrt[3]{4},\sqrt[4]{4}$ (iii) $\sqrt[3]{2},\sqrt[9]{4},\sqrt[6]{3}$

1.4 Rationalization of Surds

Rationalization of Surds

When the denominator of an expression contains a term with a square root or a number under radical sign, the process of converting into an equivalent expression whose denominator is a rational number is called rationalizing the denominator.

If the product of two irrational numbers is rational, then each one is called the **rationalizing factor** of the other.

Let a and b be integers and x, y be positive integers. Then

Remark

(i) $(a+\sqrt{x})$ and $(a-\sqrt{x})$ are rationalizing factors of each other.

(ii) $(a+b\sqrt{x})$ and $(a-b\sqrt{x})$ are rationalizing factors of each other.

(iii) $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$ are rationalizing factors of each other.

(iv) $a+\sqrt{b}$ is also called the conjugate of $a-\sqrt{b}$ and $a-\sqrt{b}$ is called the conjugate of $a+\sqrt{b}$.

(v) For rationalizing the denominator of a number, we multiply its numerator and denominator by its rationalizing factor.

Example 1.12

Rationalize the denominator of $\frac{2}{\sqrt{3}}$

Solution Multiplying the numerator and denominator of the given number by $\sqrt{3}$, we get

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Example 1.13

Rationalize the denominator of $\frac{1}{5+\sqrt{3}}$

Solution The denominator is $5+\sqrt{3}$. Its conjugate is $5-\sqrt{3}$ or the rationalizing factor is $5-\sqrt{3}$.

$$\frac{1}{5+\sqrt{3}} = \frac{1}{5+\sqrt{3}} \times \frac{5-\sqrt{3}}{5-\sqrt{3}}$$

$$= \frac{5-\sqrt{3}}{5^2-(\sqrt{3})^2} = \frac{5-\sqrt{3}}{25-3} = \frac{5-\sqrt{3}}{22}$$

Example 1.14

Simplify $\frac{1}{8-2\sqrt{5}}$ by rationalizing the denominator.

Solution Here the denominator is $8-2\sqrt{5}$. The rationalizing factor is $8+2\sqrt{5}$

$$\frac{1}{8-2\sqrt{5}} = \frac{1}{8-2\sqrt{5}} \times \frac{8+2\sqrt{5}}{8+2\sqrt{5}}$$

$$= \frac{8+2\sqrt{5}}{8^2-(2\sqrt{5})^2} \quad = \frac{8+2\sqrt{5}}{64-20}$$

$$= \frac{8+2\sqrt{5}}{44} \quad = \frac{2(4+\sqrt{5})}{44} = \frac{4+\sqrt{5}}{22}$$

Example 1.15

Simplify $\frac{1}{\sqrt{3}+\sqrt{5}}$ by rationalizing the denominator.

Solution Here the denominator is $\sqrt{3}+\sqrt{5}$. So, the rationalizing factor is $\sqrt{3}-\sqrt{5}$

$$\frac{1}{\sqrt{3}+\sqrt{5}} = \frac{1}{\sqrt{3}+\sqrt{5}} \times \frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}}$$

$$= \frac{\sqrt{3}-\sqrt{5}}{(\sqrt{3})^2-(\sqrt{5})^2} \quad = \frac{\sqrt{3}-\sqrt{5}}{3-5}$$

$$= \frac{\sqrt{3}-\sqrt{5}}{-2} \quad = \frac{\sqrt{5}-\sqrt{3}}{2}$$

Example 1.16

If $\frac{\sqrt{7}-1}{\sqrt{7}+1} + \frac{\sqrt{7}+1}{\sqrt{7}-1} = a + b\sqrt{7}$, find the values of a and b.

Solution $\frac{\sqrt{7}-1}{\sqrt{7}+1} + \frac{\sqrt{7}+1}{\sqrt{7}-1} = \frac{\sqrt{7}-1}{\sqrt{7}+1} \times \frac{\sqrt{7}-1}{\sqrt{7}-1} + \frac{\sqrt{7}+1}{\sqrt{7}-1} \times \frac{\sqrt{7}+1}{\sqrt{7}+1}$

$$= \frac{(\sqrt{7}-1)^2}{(\sqrt{7})^2-1} + \frac{(\sqrt{7}+1)^2}{(\sqrt{7})^2-1}$$

$$= \frac{7+1-2\sqrt{7}}{7-1} + \frac{7+1+2\sqrt{7}}{7-1}$$

$$= \frac{8-2\sqrt{7}}{6} + \frac{8+2\sqrt{7}}{6}$$

$$= \frac{8-2\sqrt{7}+8+2\sqrt{7}}{6}$$

$$= \frac{16}{6} = \frac{8}{3} + 0\sqrt{7}$$

$$\therefore \frac{8}{3} + 0\sqrt{7} = a + b\sqrt{7} \Longrightarrow a = \frac{8}{3},\ b = 0.$$

Exmaple 1.17

If $x = 1 + \sqrt{2}$, find $\left(x - \frac{1}{x}\right)^2$

Solution $x = 1 + \sqrt{2}$

$$\Longrightarrow \quad \frac{1}{x} = \frac{1}{1+\sqrt{2}}$$

$$= \frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}}$$

$$= \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1} = -(1-\sqrt{2})$$

$$\therefore\ x - \frac{1}{x} = (1+\sqrt{2}) - \{-(1-\sqrt{2})\}$$

$$= 1 + \sqrt{2} + 1 - \sqrt{2} = 2$$

Hence, $\left(x - \frac{1}{x}\right)^2 = 2^2 = 4.$

- If 'a' is a positive rational number and n is a positive integer such that $\sqrt[n]{a}$ is an irrational number, then $\sqrt[n]{a}$ is called a 'surd' or a 'radical'.
- For positive integers m, n and positive rational numbers a, b we have

  (i) $\left(\sqrt[n]{a}\right)^n = a = \sqrt[n]{a^n}$ (ii) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

  (iii) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$ (iv) $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

- When the denominator of an expression contains a term with a square root or a number under radical sign, the process of converting to an equivalent expression whose denominator is a rational number is called rationalizing the denominator.
- If the product of two irrational numbers is rational, then each one is called the rationalizing factor of the other.
- If a and b are any two positive integers, there exist two non-negative integers q and r such that $a = bq + r$, $0 \le r < b$. (Division Algorithm)

Exercise 1.4

Multiple Choice Questions

1. Which one of the following is not a surd?

   (A) $\sqrt[3]{8}$ (B) $\sqrt[3]{30}$ (C) $\sqrt[5]{4}$ (D) $\sqrt[8]{3}$

2. The simplest form of $\sqrt{50}$ is

   (A) $5\sqrt{10}$ (B) $5\sqrt{2}$ (C) $10\sqrt{5}$ (D) $25\sqrt{2}$

3. $\sqrt[4]{11}$ is equal to

   (A) $\sqrt[8]{11^2}$ (B) $\sqrt[8]{11^4}$ (C) $\sqrt[8]{11^8}$ (D) $\sqrt[8]{11^6}$

4. $\frac{2}{\sqrt{2}}$ is equal to

   (A) $2\sqrt{2}$ (B) $\sqrt{2}$ (C) $\frac{\sqrt{2}}{2}$ (D) 2

5. The ratioanlising factor of $\frac{5}{\sqrt[3]{3}}$ is

   (A) $\sqrt[3]{6}$ (B) $\sqrt[3]{3}$ (C) $\sqrt[3]{9}$ (D) $\sqrt[3]{27}$

Key Concept **Scientific Notation**

A number N is in *scientific notation* when it is expressed as the product of a decimal number between 1 and 10 and some integral power of 10.

$$N = a \times 10^n, \text{ where } 1 \leq a < 10 \text{ and } n \text{ is an integer.}$$

To transform numbers from decimal notation to scientific notation, the laws of exponents form the basis for calculations using powers. Let m and n be natural numbers and a is a real number. The laws of exponents are given below:

(i) $a^m \times a^n = a^{m+n}$ (Product law)

(ii) $\frac{a^m}{a^n} = a^{m-n}$ (Quotient law)

(iii) $(a^m)^n = a^{mn}$ (Power law)

(iv) $a^m \times b^m = (a \times b)^m$ (Combination law)

For $a \neq 0$, we define $a^{-m} = \frac{1}{a^m}$, and $a^0 = 1$.

Example 2.1

Express 9781 in scientific notation.

Solution In integers, the decimal point at the end is usually omitted.

$$9\;7\;8\;1\,.$$

$$\;3\;\;2\;\;1$$

The decimal point is to be moved 3 places to the left of its original position. So the power of 10 is 3.

$\therefore\ 9781 = 9.781 \times 10^{3}$

Example 2.2

Express $0 \cdot 000432078$ in scientific notation.

Solution $0\;.\;0\;0\;0\;4\;3\;2\;0\;7\;8$

$$1\;\;2\;\;3\;\;4$$

The decimal point is to be moved four places to the right of its original position. So the power of 10 is – 4

$\therefore\ 0.000432078 = 4.32078 \times 10^{-4}$

2.2.1 Multiplication and Division in Scientific Notation

One can find the product or quotient of very large(googolplex) or very small numbers easily in scientific notion.

Example 2.5

Write the following in scientific notation.

(i) $(4000000)^{3}$ (ii) $(5000)^{4} \times (200)^{3}$

(iii) $(0.00003)^{5}$ (iv) $(2000)^{2} \div (0.0001)^{4}$

Solution

(i) First we write the number (within the brackets) in scientific notation.

$$4000000 = 4.0 \times 10^{6}$$

Now, raising to the power 3 on both sides we get,

$\therefore (4000000)^3 = (4.0 \times 10^6)^3 = (4.0)^3 \times (10^6)^3$

$= 64 \times 10^{18} \quad = 6.4 \times 10^1 \times 10^{18} = 6.4 \times 10^{19}$

(ii) In scientific notation,

$5000 = 5.0 \times 10^3$ and $200 = 2.0 \times 10^2$.

$\therefore (5000)^4 \times (200)^3 = (5.0 \times 10^3)^4 \times (2.0 \times 10^2)^3$

$= (5.0)^4 \times (10^3)^4 \times (2.0)^3 \times (10^2)^3$

$= 625 \times 10^{12} \times 8 \times 10^6 = 5000 \times 10^{18}$

$= 5.0 \times 10^3 \times 10^{18} = 5.0 \times 10^{21}$

(iii) In scientific notation, $0.00003 = 3.0 \times 10^{-5}$

$\therefore (0.00003)^5 = (3.0 \times 10^{-5})^5 = (3.0)^5 \times (10^{-5})^5$

$= 243 \times 10^{-25} \quad = 2.43 \times 10^2 \times 10^{-25} = 2.43 \times 10^{-23}$

(iv) In scientific notation,

$2000 = 2.0 \times 10^3$ and $0.0001 = 1.0 \times 10^{-4}$

$$\therefore (2000)^2 \div (0.0001)^4 = \frac{(2.0 \times 10^3)^2}{(1.0 \times 10^{-4})^4} = \frac{(2.0)^2 \times (10^3)^2}{(1.0)^4 \times (10^{-4})^4}$$

$$= \frac{4 \times 10^6}{10^{-16}} = 4.0 \times 10^{6-(-16)} = 4.0 \times 10^{22}$$

Exercise 2.1

1. Represent the following numbers in the scientific notation.

   (i) 749300000000 (ii) 13000000 (iii) 105003

   (iv) 543600000000000 (v) 0.0096 (vi) 0.0000013307

   (vii) 0.0000000022 (viii) 0.0000000000009

2. Write the following numbers in decimal form.

   (i) 3.25×10^{-6} (ii) 4.134×10^{-4} (iii) 4.134×10^{4}

   (iv) 1.86×10^{7} (v) 9.87×10^{9} (vi) 1.432×10^{-9}

3. Represent the following numbers in scientific notation.

   (i) $(1000)^2 \times (20)^6$ (ii) $(1500)^3 \times (0.0001)^2$

   (iii) $(16000)^3 \div (200)^4$ (iv) $(0.003)^7 \times (0.0002)^5 \div (0.001)^3$

   (v) $(11000)^3 \times (0.003)^2 \div (30000)$

Key Concept	**Sector**
A sector is the part of a circle enclosed by any two radii of the circle and their intercepted arc.	

4.2.1 Central Angle or Sector Angle of a Sector

Key Concept	**Central Angle**
Central Angle is the angle subtended by the arc of the sector at the centre of the circle in which the sector forms a part.	

In fig.4.2, the angle subtended by the arc $\overset{\frown}{PQ}$ at the centre is θ. So the central angle of the sector POQ is θ.

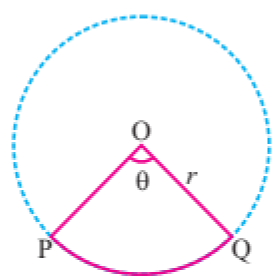


Fig. 4.2

For example,

1. A semi- circle is a sector whose central angle is 180°.
2. A quadrant of a circle is a sector whose central angle is 90°.

4.2.2 Length of Arc (Arc Length) of a Sector

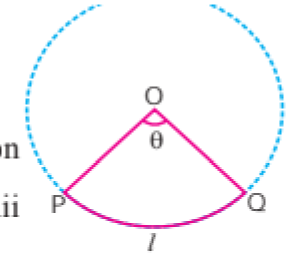


Fig. 4.3

In fig.4.3, arc length of a sector *POQ* is the length of the portion on the circumference of the circle intercepted between the bounding radii (*OP* and *OQ*) and is denoted by *l*.

For example,

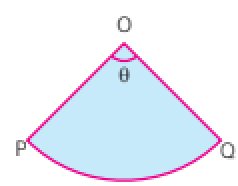
1. Length of arc of a circle is its circumference. i.e., $l=2\pi r$ units, where r is the radius.
2. Length of arc of a semicircle is $l=2\pi r\times\frac{180^\circ}{360^\circ}=\pi r$ units, where r is the radius and the central angle is 180°.
3. Length of arc of a quadrant of a circle is $l=2\pi r\times\frac{90^\circ}{360^\circ}=\frac{\pi r}{2}$ units, where r is the radius and the central angle is 90°.

Key Concept	Length of Arc
If θ is the central angle and r is the radius of a sector, then its arc length is given by $l=\frac{\theta}{360^\circ}\times 2\pi r$ units	

4.2.3 Area of a Sector

Area of a sector is the region bounded by the bounding radii and the arc of the sector.

For Example,



1. Area of a circle is πr^2 square units.
2. Area of a semicircle is $\frac{\pi r^2}{2}$ square units.
3. Area of a quadrant of a circle is $\frac{\pi r^2}{4}$ square units.

Key Concept	Area of a Sector
If θ is the central angle and r is the radius of a sector, then the area of the sector is $\frac{\theta}{360^\circ}\times\pi r^2$ square units.	

Let us find the relationship between area of a sector, its arc length l and radius r.

$$\text{Area} = \frac{\theta}{360^\circ} \times \pi r^2$$

$$= \frac{\theta}{360^\circ} \times \frac{2\pi r}{2} \times r$$

$$= \frac{1}{2} \times \left(\frac{\theta}{360^\circ} \times 2\pi r\right) \times r$$

$$= \frac{1}{2} \times lr$$

Area of sector $= \frac{lr}{2}$ square units.

4.2.4 Perimeter of a Sector

The perimeter of a sector is the sum of the lengths of all its boundaries. Thus, perimeter of a sector is $l + 2r$ units.

Key Concept **Perimeter of a Sector**

If l is the arc length and r is the radius of a sector, then its perimeter P is given by the formula $P = l + 2r$ units.

For example,

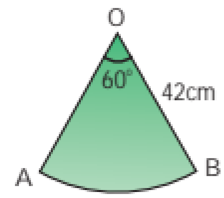
1. Perimeter of a semicircle is $(\pi + 2)r$ units.
2. Perimeter of a quadrant of a circle is $\left(\frac{\pi}{2} + 2\right)r$ units.

Note

1. Length of an arc and area of a sector are proportional to the central angle.
2. As π is an irrational number, we use its approximate val1ue $\frac{22}{7}$ or 3.14 in our calculations.

Example 4.1

The radius of a sector is 42 cm and its sector angle is 60°. Find its arc length, area and perimeter.



Solution Given that radius $r = 42$ cm and $\theta = 60^\circ$. Therefore,

$$\text{length of arc } l = \frac{\theta}{360^\circ}\times 2\pi r$$

$$= \frac{60^\circ}{360^\circ}\times 2\times\frac{22}{7}\times 42 = 44 \text{ cm.}$$

$$\text{Area of the sector} = \frac{lr}{2} = \frac{44\times 42}{2} = 924 \text{ cm}^2.$$

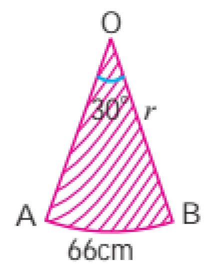
$$\text{Perimeter} = l + 2r$$

$$= 44 + 2(42) = 128 \text{ cm.}$$

Example 4.2

The arc length of a sector is 66 cm and the central angle is 30°. Find its radius.

Solution Given that $\theta = 30^\circ$ and $l = 66$ cm. So,



$$\frac{\theta}{360^\circ}\times 2\pi r = l$$

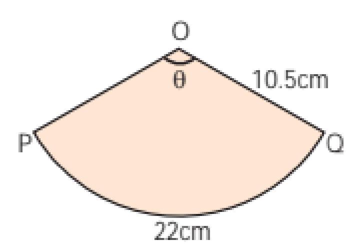
$$\text{i. e., } \frac{30^\circ}{360^\circ}\times 2\times\frac{22}{7}\times r = 66$$

$$\therefore r = 66\times\frac{360^\circ}{30^\circ}\times\frac{1}{2}\times\frac{7}{22} = 126 \text{ cm}$$

Example 4.3

The length of arc of a sector is 22 cm and its radius is 10.5 cm. Find its central angle.

Solution Given that $r = 10.5$ cm and $l = 22$ cm.



$$\frac{\theta}{360^\circ}\times 2\pi r = l$$

$$\text{i. e., } \frac{\theta}{360^\circ}\times 2\times\frac{22}{7}\times 10.5 = 22$$

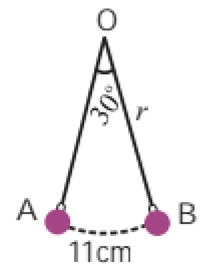
$$\therefore \theta = 22\times 360^\circ\times\frac{1}{2}\times\frac{7}{22}\times\frac{1}{10.5} = 120^\circ$$

Example 4.4

A pendulum swings through an angle of 30° and describes an arc length of 11 cm. Find the length of the pendulum.

Solution If the pendulum swings once, then it forms a sector and the radius of the sector is the length of the pendulum. So,

$$\theta = 30^\circ,\ l = 11 \text{ cm}$$



Using the formula $\frac{\theta}{360^\circ} \times 2\pi r = l$, we have

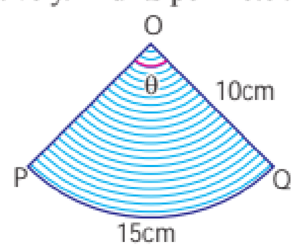
$$\frac{30^\circ}{360^\circ} \times 2 \times \frac{22}{7} \times r = 11$$

$$\therefore\ r = 11 \times \frac{360^\circ}{30^\circ} \times \frac{1}{2} \times \frac{7}{22} = 21 \text{ cm}$$

Example 4.5

The radius and length of arc of a sector are 10 cm and 15 cm respectively.Find its perimeter.

Solution Given that $r = 10$ cm, $l = 15$ cm



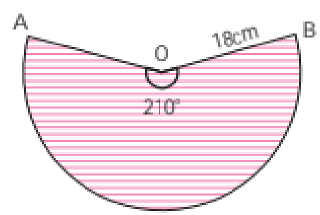
Perimeter of the sector $= l + 2r = 15 + 2(10)$

$= 15 + 20 = 35$ cm

Example 4.6

Find the perimeter of a sector whose radius and central angle are 18 cm and 210° respectively.

Solution Given that $r = 18$ cm, $\theta = 210^\circ$. Hence,



$$l = \frac{\theta}{360^\circ} \times 2\pi r$$

$$= \frac{210^\circ}{360^\circ} \times 2 \times \frac{22}{7} \times 18 = 66 \text{ cm}$$

$\therefore$ Perimeter of the sector $= l + 2r = 66 + 2(18) = 66 + 36 = 102$ cm

Example 4.7

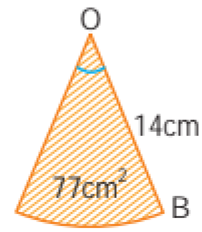
The area of a sector of a circle of radius 14 cm is 77 cm². Find its central angle.

Solution Given that $r = 14$ cm, area $= 77$ cm²

$$\frac{\theta}{360^\circ} \times \pi r^2 = \text{Area of the sector}$$

$$\frac{\theta}{360^\circ} \times \frac{22}{7} \times 14 \times 14 = 77$$

$$\therefore\ \theta = \frac{77 \times 360^\circ \times 7}{22 \times 14 \times 14} = 45^\circ$$



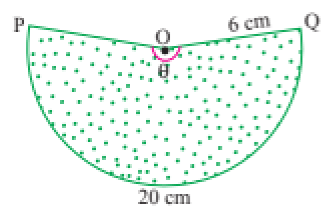
Example 4.8

Calculate the area of a sector whose radius and arc length are 6 cm and 20 cm respectively.

Solution Given that $r = 6$ cm, $l = 20$ cm

$$\text{Area} = \frac{lr}{2} \text{ square units}$$

$$= \frac{20 \times 6}{2} = 60 \text{ cm}^2$$



Example 4.9

If the perimeter and radius of a sector are 38 cm and 9 cm respectively, find its area.

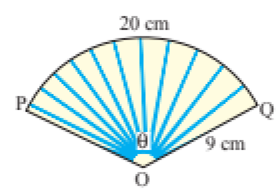
Solution Given, $r = 9$ cm, perimeter $= 38$ cm

$$\text{Perimeter} = l + 2r = 38$$

i.e., $l + 18 = 38$

$$l = 38 - 18 = 20 \text{ cm}$$

$$\therefore \text{Area} = \frac{lr}{2} = \frac{20 \times 9}{2} = 90 \text{ cm}^2$$



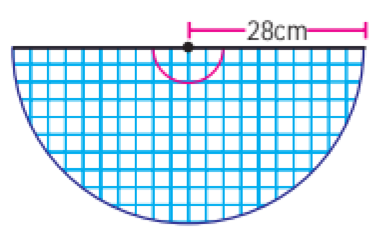
Example 4.10

Find the perimeter and area of a semicircle of radius 28 cm.

Solution Given, $r = 28$ cm

$$\text{Perimeter} = (\pi + 2)\ r = (\frac{22}{7} + 2)\ 28 = 144 \text{ cm}$$

$$\text{Area} = \frac{\pi r^2}{2} = \frac{22}{7} \times \frac{28 \times 28}{2} = 1232 \text{ cm}^2$$



Example 4.11

Find the radius, central angle and perimeter of a sector whose arc length and area are 27.5 cm and 618.75 cm² respectively.

Solution Given that $l = 27.5$ cm and Area $= 618.75$ cm². So,

$$\text{Area} = \frac{lr}{2} = 618.75 \text{ cm}^2$$

$$\text{i.e. } \frac{27.5 \times r}{2} = 618.75$$

$$\therefore r = 45 \text{ cm}$$

Hence, perimeter is $l + 2r = 27.5 + 2(45) = 117.5$cm

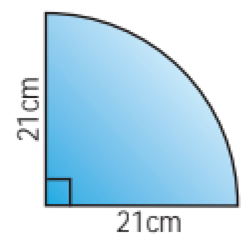
Now, arc length is given by $\frac{\theta}{360^\circ} \times 2\pi r = l$

$$\text{i.e. } \frac{\theta}{360^\circ} \times 2 \times \frac{22}{7} \times 45 = 27.5$$

$$\therefore \theta = 35^\circ$$

Example 4.12

Calculate the area and perimeter of a quadrant of a circle of radius 21 cm.



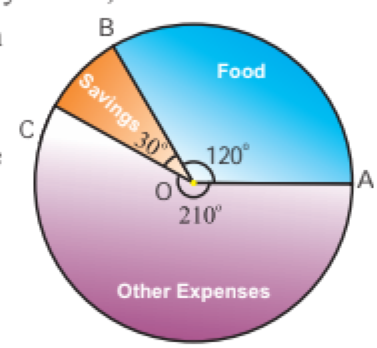
Solution Given that $r = 21$ cm, $\theta = 90^\circ$

$$\text{Perimeter} = \left(\frac{\pi}{2} + 2\right)r = \left(\frac{22}{7 \times 2} + 2\right) \times 21 = 75 \text{ cm}$$

$$\text{Area} = \frac{\pi r^2}{4} = \frac{22}{7 \times 4} \times 21 \times 21 = 346.5 \text{ cm}^2$$

Example 4.13

Monthly expenditure of a person whose monthly salary is ₹ 9,000 is shown in the adjoining figure. Find the amount he has (i) spent for food (ii) in his savings



Solution Let ₹ 9,000 be represented by the area of the circle, i. e., $\pi r^2 = 9000$

(i) Area of sector $AOB = \frac{\theta}{360^\circ} \times \pi r^2$

$$= \frac{120^\circ}{360^\circ} \times 9000 = 3,000$$

Amount spent for food is ₹ 3,000.

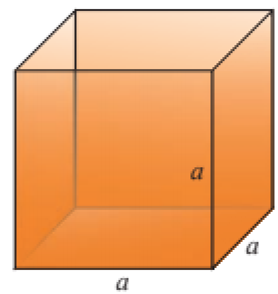
(ii) Area of sector $BOC = \frac{\theta}{360^\circ} \times \pi r^2$

$$= \frac{30^\circ}{360^\circ} \times 9,000 = 750$$

Amount saved in savings is ₹ 750.

4.3.1 Surface Area of a Cube

The sum of the areas of all the six equal faces is called the *Total Surface Area* (T.S.A) of the cube.



In the adjoining figure, let the side of the cube measure a units. Then the area of each face of the cube is a^2 square units. Hence, the total surface area is $6a^2$ square units.

In a cube, if we don't consider the top and bottom faces, the remaining area is called the *Lateral Surface Area* (L.S.A). Hence, the lateral surface area of the cube is $4a^2$ square units.

Key Concept **Surface Area of Cube**

Let the side of a cube be a units. Then:

(i) The Total Surface Area (T.S.A) = $6a^2$ square units.

(ii) The Lateral Surface Area (L.S.A) = $4a^2$ square units.

4.3.2 Volume of a Cube

Key Concept **Volume of Cube**

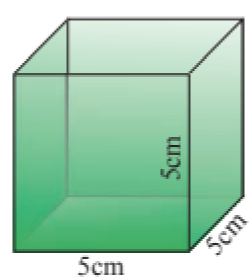
If the side of a cube is a units, then its volume V is given by the formula

$$V = a^3 \text{ cubic units}$$

Note Volume can also be defined as the number of unit cubes required to fill the entire cube.

Example 4.16

Find the L.S.A, T.S.A and volume of a cube of side 5 cm.



Solution L.S.A $= 4a^2 = 4(5^2) = 100$ sq. cm

T.S.A $= 6a^2 = 6\,(5^2) = 150$ sq. cm

Volume $= a^3 = 5^3 = 125$ cm^3

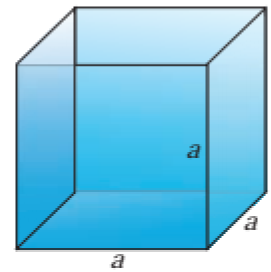
Example: 4.17

Find the length of the side of a cube whose total surface area is 216 square cm.

Solution Let a be the side of the cube. Given that T.S.A = 216 sq. cm

i. e., $6a^2 = 216 \implies a^2 = \frac{216}{6} = 36$

$\therefore \quad a = \sqrt{36} = 6$ cm



Example 4.18

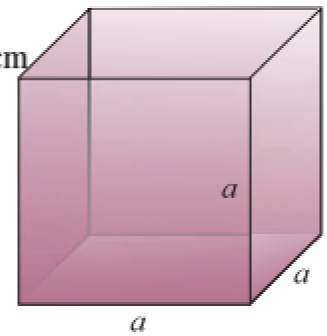
A cube has a total surface area of 384 sq. cm. Find its volume.

Solution Let a be the side of the cube. Given that T.S.A = 384 sq. cm

$6a^2 = 384 \implies a^2 = \frac{384}{6} = 64$

$\therefore \; a = \sqrt{64} = 8$ cm

Hence, Volume $= a^3 = 8^3 = 512$ cm^3



Example 4.19

A cubical tank can hold 27,000 litres of water. Find the dimension of its side.

Solution Let a be the side of the cubical tank. Volume of the tank is 27,000 litres. So,

$V = a^3 = \frac{27,000}{1,000} m^3 = 27\, m^3 \qquad \therefore a = \sqrt[3]{27} = 3\ m$

4.4 Cuboids

A cuboid is a three dimensional solid having six rectangular faces.

Example: Bricks, Books etc.,

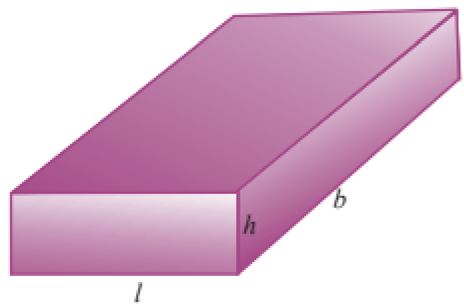
4.4.1 Surface Area of a Cuboid

Let l, b and h be the length, breadth and height of a cuboid respectively. To find the total surface area, we split the faces into three pairs.

(i) The total area of the front and back faces is $lh + lh = 2lh$ square units.

(ii) The total area of the side faces is $bh + bh = 2bh$ square units.

(iii) The total area of the top and bottom faces is $lb + lb = 2lb$ square units.



The Lateral Surface Area (L.S.A) = $2(l + b)h$ square units.

The Total Surface Area (T.S.A) = $2(lb + bh + lh)$ square units.

Key Concept — Surface Area of a Cuboid

Let l, b and h be the length, breadth and height of a cuboid respectively. Then

(i) The Lateral Surface Area (L.S.A) = $2(l + b)h$ square units

(ii) The Total Surface Area (T.S.A) = $2(lb + bh + lh)$ sq. units

Note L.S.A. is also equal to the product of the perimeter of the base and the height.

4.4.2 Volume of a Cuboid

Key Concept — Volume of a Cuboid

If the length, breadth and height of a cuboid are l, b and h respectively, then the volume V of the cuboid is given by the formula

$$V = l \times b \times h \text{ cu. units}$$

Example: 4.20

Find the total surface area of a cuboid whose length, breadth and height are 20 cm, 12 cm and 9 cm respectively.

Solution

Given that $l = 20$ cm, $b = 12$ cm, $h = 9$ cm

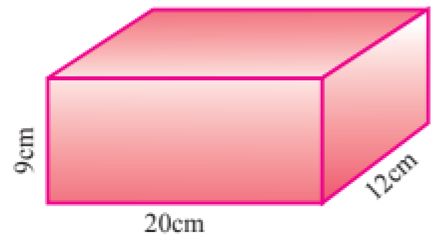
$\therefore$ T.S.A $= 2(lb + bh + lh)$

$= 2[(20\times12)+(12\times9)+(20\times9)]$

$= 2(240+108+180)$

$= 2\times528$

$= 1056\ cm^2$



Example: 4.21

Find the L.S.A of a cuboid whose dimensions are given by $3m\times5m\times4m$.

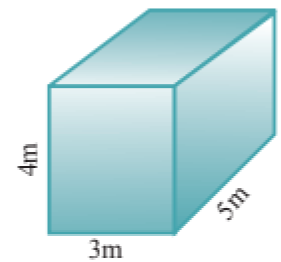
Solution Given that $l = 3$ m, b = 5 m, h = 4 m

L.S.A $= 2(l+b)h$

$= 2\times(3+5)\times4$

$= 2\times8\times4$

$= 64$ sq. m



Example: 4.22

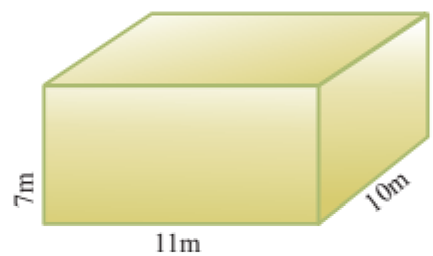
Find the volume of a cuboid whose dimensions are given by 11 m, 10 m and 7 m.

Solution Given that $l = 11$ m, b = 10 m, h = 7 m

volume $= lbh$

$= 11\times10\times7$

$= 770$ cu.m.



Example: 4.23

Two cubes each of volume 216 cm^3 are joined to form a cuboid as shown in the figure. Find the T.S.A of the resulting cuboid.

Solution Let the side of each cube be a. Then $a^3 = 216$

$\therefore a = \sqrt[3]{216} = 6$ cm

Now the two cubes of side 6 cm are joined to form a cuboid. So,

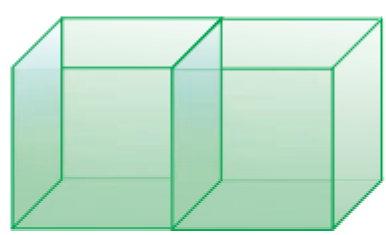
$\therefore l = 6+6 = 12$ cm, $b = 6$ cm, $h = 6$ cm

$\therefore$ T.S.A $= 2(lb+bh+lh)$

$= 2[(12\times6)+(6\times6)+(12\times6)]$

$= 2[72+36+72]$

$= 2\times180 = 360\ cm^2$



- A sector is the part of a circle enclosed by any two radii of the circle and their intercepted arc.
- Central Angle is the angle subtended by the arc of the sector at the centre of the circle in which the sector forms a part.
- If θ is the central angle and r is the radius of a sector, then its arc length is given by $l = \frac{\theta}{360} \times 2\pi r$ units
- If θ is the central angle and r is the radius of a sector, then the area of the sector is $\frac{\theta}{360} \times \pi r^2$ square units.
- If l is the arc length and r is the radius of a sector, then its perimeter P is given by the formula $P = l + 2r$ units.

- Let the side of a cube be a units. Then:
  (i) The Total Surface Area (T.S.A) = $6a^2$ square units.
  (ii) The Lateral Surface Area (L.S.A) = $4a^2$ square units.
- If the side of a cube is a units, then its volume V is given by the formula, $V = a^3$ cubic units
- Let l, b and h be the length, breadth and height of a cuboid respectively. Then:
  (i) The Lateral Surface Area (L.S.A) = $2(l + b)h$ squre units
  (ii) The Total Surface Area (T.S.A) = $2(lb + bh + lh)$ sq. units
- If the length, breadth and height of a cuboid are l, b and h respectively, then the volume V of the cuboid is given by the formula $V = l \times b \times h$ cu. units

Samacheer Kalvi Maths

10th Std

(i) $2, 4, 6, 8, \cdots, 2010.$ (finite number of terms)

(ii) $1, -1, 1, -1, 1, -1, 1, \cdots .$ (terms just keep oscillating between 1 and -1)

(iii) $\pi, \pi, \pi, \pi, \pi.$ (terms are same; such sequences are constant sequences)

(iv) $2, 3, 5, 7, 11, 13, 17, 19, 23, \cdots .$ (list of all prime numbers)

(v) $0.3, 0.33, 0.333, 0.3333, 0.33333, \cdots .$ (infinite number of terms)

(vi) $S = \{a_n\}_1^\infty$ where $a_n = 1$ or 0 according to the outcome head or tail in the n^{th} toss of a coin.

Example 2.1

Write the first three terms in a sequence whose n^{th} term is given by

$$c_n = \frac{n(n+1)(2n+1)}{6}, \ \forall\, n \in \mathbb{N}$$

Solution Here,

$$c_n = \frac{n(n+1)(2n+1)}{6}, \ \forall\, n \in \mathbb{N}$$

For $n = 1$, $c_1 = \frac{1(1+1)(2(1)+1)}{6} = 1.$

For $n = 2$, $c_2 = \frac{2(2+1)(4+1)}{6} = \frac{2(3)(5)}{6} = 5.$

Finally $n = 3$, $c_3 = \frac{3(3+1)(7)}{6} = \frac{(3)(4)(7)}{6} = 14.$

Hence, the first three terms of the sequence are 1, 5, and 14.

In the above example, we were given a formula for the general term and were able to find any particular term directly. In the following example, we shall see another way of generating a sequence.

Example 2.13

An amount ₹500 is deposited in a bank which pays annual interest at the rate of 10% compounded annually. What will be the value of this deposit at the end of 10^{th} year?

Solution

The principal is ₹500. So, the interest for this principal for one year is $500\left(\frac{10}{100}\right) = 50$.

Thus, the principal for the 2nd year = Principal for 1st year + Interest

$$= 500 + 500\left(\frac{10}{100}\right) = 500\left(1 + \frac{10}{100}\right)$$

Now, the interest for the second year $= \left(500\left(1+\frac{10}{100}\right)\right)\left(\frac{10}{100}\right)$.

So, the principal for the third year $= 500\left(1+\frac{10}{100}\right) + 500\left(1+\frac{10}{100}\right)\frac{10}{100}$

$$= 500\left(1+\frac{10}{100}\right)^2$$

Continuing in this way we see that the principal for the n^{th} year $\left.\right\} = 500\left(1+\frac{10}{100}\right)^{n-1}$.

The amount at the end of $(n-1)^{th}$ year = Principal for the n^{th} year.

Thus, the amount in the account at the end of n^{th} year.

$$= 500\left(1+\frac{10}{100}\right)^{n-1} + 500\left(1+\frac{10}{100}\right)^{n-1}\left(\frac{10}{100}\right) = 500\left(\frac{11}{10}\right)^n.$$

The amount in the account at the end of 10^{th} year

$$= ₹\ 500\left(1+\frac{10}{100}\right)^{10} = ₹\ 500\left(\frac{11}{10}\right)^{10}.$$

Remarks

By using the above method, one can derive a formula for finding the total amount for compound interest problems. Derive the formula:

$$A = P(1+i)^n$$

where A is the amount, P is the principal, $i = \frac{r}{100}$, r is the annual interest rate and n is the number of years.

Example 2.16

Find the sum of the arithmetic series $5 + 11 + 17 + \cdots + 95$.

Solution Given that the series $5 + 11 + 17 + \cdots + 95$ is an arithmetic series.

Note that $a = 5$, $d = 11 - 5 = 6$, $l = 95$.

Now, $n = \frac{l-a}{d} + 1$

$= \frac{95-5}{6} + 1 = \frac{90}{6} + 1 = 16.$

Hence, the sum $S_n = \frac{n}{2}[l + a]$

$S_{16} = \frac{16}{2}[95 + 5] = 8(100) = 800.$

Example 2.17

Find the sum of the first $2n$ terms of the following series.

$1^2 - 2^2 + 3^2 - 4^2 + \dots$.

Solution We want to find $1^2 - 2^2 + 3^2 - 4^2 + \cdots$ to $2n$ terms

$= 1 - 4 + 9 - 16 + 25 - \cdots$ to $2n$ terms

$= (1-4) + (9-16) + (25-36) + \cdots$ to n terms. (after grouping)

$= -3 + (-7) + (-11) + \cdots$ n terms

Now, the above series is in an A.P. with first term $a = -3$ and common difference $d = -4$

Therefore, the required sum $= \frac{n}{2}[2a + (n-1)d]$

$= \frac{n}{2}[2(-3) + (n-1)(-4)]$

$= \frac{n}{2}[-6 - 4n + 4] = \frac{n}{2}[-4n - 2]$

$= \frac{-2n}{2}(2n+1) = -n(2n+1).$

(i) The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(ii) The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k-1) = n^2$.

(iii) The sum of first n odd natural numbers (when the last term l is given) is

$$1+3+5+\cdots+l = \left(\frac{l+1}{2}\right)^2.$$

(iv) The sum of squares of first n natural numbers,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(v) The sum of cubes of the first n natural numbers,

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Example 2.29

Find the sum of the following series

(i) $26+27+28+\cdots+60$ (ii) $1+3+5+\cdots$ to 25 terms (iii) $31+33+\cdots+53$.

Solution

(i) We have $26+27+28+\cdots+60 = (1+2+3+\cdots+60)-(1+2+3+\cdots+25)$

$$= \sum_{1}^{60} n - \sum_{1}^{25} n$$

$$= \frac{60(60+1)}{2} - \frac{25(25+1)}{2}$$

$$= (30\times 61)-(25\times 13) = 1830-325 = 1505.$$

(ii) Here, $n = 25$

$\therefore$ $1+3+5+\cdots$ to 25 terms $= 25^2$ $\left(\sum_{k=1}^{n}(2k-1) = n^2\right)$

$= 625.$

(iii) $31+33+\cdots+53$

$$= (1+3+5+\cdots+53)-(1+3+5+\cdots+29)$$

$$= \left(\frac{53+1}{2}\right)^2 - \left(\frac{29+1}{2}\right)^2 \qquad \left(1+3+5+\cdots+l = \left(\frac{l+1}{2}\right)^2\right)$$

$$= 27^2-15^2 = 504.$$

Example 2.30

Find the sum of the following series

(i) $1^2 + 2^2 + 3^2 + \cdots + 25^2$ (ii) $12^2 + 13^2 + 14^2 + \cdots + 35^2$

(iii) $1^2 + 3^2 + 5^2 + \cdots + 51^2$.

Solution

(i) Now, $1^2 + 2^2 + 3^2 + \cdots + 25^2 = \sum_{1}^{25} n^2$

$$= \frac{25(25+1)(50+1)}{6} \qquad \left(\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \right)$$

$$= \frac{(25)(26)(51)}{6}$$

$\therefore$ $1^2 + 2^2 + 3^2 + \cdots + 25^2 = 5525$.

(ii) Now, $12^2 + 13^2 + 14^2 + \cdots + 35^2$

$$= (1^2 + 2^2 + 3^2 + \cdots + 35^2) - (1^2 + 2^2 + 3^2 + \cdots + 11^2)$$

$$= \sum_{1}^{35} n^2 - \sum_{1}^{11} n^2$$

$$= \frac{35(35+1)(70+1)}{6} - \frac{11(12)(23)}{6}$$

$$= \frac{(35)(36)(71)}{6} - \frac{(11)(12)(23)}{6}$$

$= 14910 - 506 = 14404.$

(iii) Now, $1^2 + 3^2 + 5^2 + \cdots + 51^2$

$$= (1^2 + 2^2 + 3^2 + \cdots + 51^2) - (2^2 + 4^2 + 6^2 + \cdots 50^2)$$

$$= \sum_{1}^{51} n^2 - 2^2[1^2 + 2^2 + 3^2 + \cdots + 25^2]$$

$$= \sum_{1}^{51} n^2 - 4\sum_{1}^{25} n^2$$

$$= \frac{51(51+1)(102+1)}{6} - 4 \times \frac{25(25+1)(50+1)}{6}$$

$$= \frac{(51)(52)(103)}{6} - 4 \times \frac{25(26)(51)}{6}$$

$= 45526 - 22100 = 23426.$

Example 2.31

Find the sum of the series.

(i) $1^3 + 2^3 + 3^3 + \cdots + 20^3$ (ii) $11^3 + 12^3 + 13^3 + \cdots + 28^3$

Solution

(i) $1^3 + 2^3 + 3^3 + \cdots 20^3 = \sum_{1}^{20} n^3$

$$= \left(\frac{20(20+1)}{2}\right)^2 \qquad \text{using } \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

$$= \left(\frac{20 \times 21}{2}\right)^2 = (210)^2 = 44100.$$

(ii) Next we consider $11^3 + 12^3 + \cdots + 28^3$

$$= (1^3 + 2^3 + 3^3 + \cdots + 28^3) - (1^3 + 2^3 + \cdots + 10^3)$$

$$= \sum_{1}^{28} n^3 - \sum_{1}^{10} n^3$$

$$= \left[\frac{28(28+1)}{2}\right]^2 - \left[\frac{10(10+1)}{2}\right]^2$$

$$= 406^2 - 55^2 = (406 + 55)(406 - 55)$$

$$= (461)(351) = 161811.$$

Example 2.33

(i) If $1 + 2 + 3 + \cdots + n = 120$, find $1^3 + 2^3 + 3^3 + \cdots n^3$.

(ii) If $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$, then find $1 + 2 + 3 + \cdots + n$.

Solution

(i) Given $1 + 2 + 3 + \cdots + n = 120$ i.e. $\frac{n(n+1)}{2} = 120$

$$\therefore \quad 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = 120^2 = 14400$$

(ii) Given $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$

$$\Longrightarrow \quad \left(\frac{n(n+1)}{2}\right)^2 = 36100 = 19 \times 19 \times 10 \times 10$$

$$\Longrightarrow \quad \frac{n(n+1)}{2} = 190$$

Thus, $1 + 2 + 3 + \cdots + n = 190$.

Example 2.34

Find the total area of 14 squares whose sides are 11 cm, 12 cm, $\cdots$, 24 cm, respectively.

Solution The areas of the squares form the series $11^2 + 12^2 + \cdots + 24^2$

Total area of 14 squares $= 11^2 + 12^2 + 13^2 + \cdots + 24^2$

$$= (1^2 + 2^2 + 3^2 + \cdots + 24^2) - (1^2 + 2^2 + 3^2 + \cdots + 10^2)$$

$$= \sum_1^{24} n^2 - \sum_1^{10} n^2$$

$$= \frac{24(24+1)(48+1)}{6} - \frac{10(10+1)(20+1)}{6}$$

$$= \frac{(24)(25)(49)}{6} - \frac{(10)(11)(21)}{6}$$

$$= 4900 - 385$$

$$= 4515 \text{ sq. cm.}$$

Points to Remember

- A sequence of real numbers is an arrangement or a list of real numbers in a specific order.
- The sequence given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots$ is called the Fibonacci sequence which is nothing but $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an arithmetic sequence if $a_{n+1} = a_n + d$, $n \in \mathbb{N}$ where d is a constant. Here a_1 is called the first term and the constant d is called the common difference.

  The formula for the general term of an A.P. is $t_n = a + (n-1)d \quad \forall \; n \in \mathbb{N}$.
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called a geometric sequence if $a_{n+1} = a_n r$, where $r \neq 0$, $n \in \mathbb{N}$ where r is a constant. Here, a_1 is the first term and the constant r is called the common ratio. The formula for the general term of a G.P. is $t_n = ar^{n-1}$, $n = 1, 2, 3, \cdots$.
- An expression of addition of terms of a sequence is called a series. If the sum consists only finite number of terms, then it is called a finite series. If the sum consists of infinite number of terms of a sequence, then it is called an infinite series.
- The sum S_n of the first n terms of an arithmetic sequence with first term a and common difference d is given by $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$, where l is the last term.

- The sum of the first n terms of a geometric series is given by

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r-1} = \dfrac{a(1-r^n)}{1-r}, & \text{if } r \neq 1 \\ na & \text{if } r = 1. \end{cases}$$

where a is the first term and r is the common ratio.

- The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
- The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k-1) = n^2$
- The sum of first n odd natural numbers (when the last term l is given) is

$$1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2.$$

- The sum of squares of first n natural numbers, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
- The sum of cubes of the first n natural numbers, $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Example 3.1

Solve $3x - 5y = -16$, $2x + 5y = 31$

Solution The given equations are

$$3x - 5y = -16 \quad (1)$$

$$2x + 5y = 31 \quad (2)$$

Note that the coefficients of y in both equations are numerically equal.

So, we can eliminate y easily.

Adding (1) and (2), we obtain an equation

$$5x = 15 \quad (3)$$

That is, $x = 3$.

Now, we substitute $x = 3$ in (1) or (2) to solve for y.

Substituting $x = 3$ in (1) we obtain, $3(3) - 5y = -16$

$$\Longrightarrow \quad y = 5.$$

Now, (3, 5) a is solution to the given system because (1) and (2) are true when $x = 3$ and $y = 5$ as from (1) and (2) we get, $3(3) - 5(5) = -16$ and $2(3) + 5(5) = 31$.

Example 3.2

The cost of 11 pencils and 3 erasers is ₹ 50 and the cost of 8 pencils and 3 erasers is ₹ 38. Find the cost of each pencil and each eraser.

Solution Let x denote the cost of a pencil in rupees and y denote the cost of an eraser in rupees.

Then according to the given information we have

$$11x + 3y = 50 \quad (1)$$

$$8x + 3y = 38 \quad (2)$$

Subtracting (2) from (1) we get, $3x = 12$ which gives $x = 4$.

Now substitute $x = 4$ in (1) to find the value of y. We get,

$$11(4) + 3y = 50 \quad \text{i.e.,} \quad y = 2.$$

Therefore, $x = 4$ and $y = 2$ is the solution of the given pair of equations.

Thus, the cost of a pencil is ₹ 4 and that of an eraser is ₹ 2.

Example 3.3

Solve by elimination method $3x + 4y = -25$, $2x - 3y = 6$

Solution The given system is

$$3x + 4y = -25 \quad (1)$$

$$2x - 3y = 6 \quad (2)$$

To eliminate the variable x, let us multiply (1) by 2 and (2) by -3 to obtain

$$(1) \times 2 \implies 6x + 8y = -50 \quad (3)$$

$$(2) \times -3 \implies -6x + 9y = -18 \quad (4)$$

Now, adding (3) and (4) we get, $17y = -68$ which gives $y = -4$

Next, substitute $y = -4$ in (1) to obtain

$$3x + 4(-4) = -25$$

That is, $x = -3$

Hence, the solution is $(-3, -4)$.

Example 3.16

(i) Prove that $x-1$ is a factor of $x^3-6x^2+11x-6$.

(ii) Prove that $x+1$ is a factor of $x^3+6x^2+11x+6$.

Solution

(i) Let $p(x) = x^3-6x^2+11x-6$.

$p(1) = 1-6+11-6 = 0$. (note that sum of the coefficients is 0)

Thus, $(x-1)$ is a factor of $p(x)$.

(ii) Let $q(x) = x^3+6x^2+11x+6$.

$q(-1) = -1+6-11+6 = 0$. Hence, $x+1$ is a factor of $q(x)$

3.5 Greatest Common Divisor (GCD) and Least Common Multiple (LCM)

3.5.1 Greatest Common Divisor (GCD)

The Highest Common Factor (HCF) or Greatest Common Divisor (GCD) of two or more algebraic expressions is the expression of highest degree which divides each of them without remainder.

Consider the simple expressions

(i) a^4, a^3, a^5, a^6 (ii) a^3b^4, ab^5c^2, a^2b^7c

In (i), note that a, a^2, a^3 are the divisors of all these expressions. Out of them, a^3 is the divisor with highest power. Therefore a^3 is the GCD of the expressions a^4, a^3, a^5, a^6.

In (ii), similarly, one can easily see that ab^4 is the GCD of a^3b^4, ab^5c^2, a^2b^7c.

If the expressions have numerical coefficients, find their greatest common divisor, and prefix it as a coefficient to the greatest common divisor of the algebraic expressions.

Let us consider a few more examples to understand the greatest common divisor.

Examples 3.19

Find the GCD of the following : (i) 90, 150, 225 (ii) $15x^4y^3z^5$, $12x^2y^7z^2$

(iii) $6(2x^2 - 3x - 2)$, $8(4x^2 + 4x + 1)$, $12(2x^2 + 7x + 3)$

Solution

(i) Let us write the numbers 90, 150 and 225 in the product of their prime factors as

$$90 = 2 \times 3 \times 3 \times 5\ ,\ 150 = 2 \times 3 \times 5 \times 5 \text{ and } 225 = 3 \times 3 \times 5 \times 5$$

From the above 3 and 5 are common prime factors of all the given numbers.

Hence the GCD $= 3 \times 5 = 15$

(ii) We shall use similar technique to find the GCD of algebraic expressions.

Now let us take the given expressions $15x^4y^3z^5$ and $12x^2y^7z^2$.

Here the common divisors of the given expressions are 3, x^2, y^3 and z^2.

Therefore, GCD $= 3 \times x^2 \times y^3 \times z^2 = 3x^2y^3z^2$

(iii) Given expressions are $6(2x^2 - 3x - 2)$, $8(4x^2 + 4x + 1)$, $12(2x^2 + 7x + 3)$

Now, GCD of 6, 8, 12 is 2

Next let us find the factors of quadratic expressions.

$$2x^2 - 3x - 2 = (2x + 1)(x - 2)$$

$$4x^2 + 4x + 1 = (2x + 1)(2x + 1)$$

$$2x^2 + 7x + 3 = (2x + 1)(x + 3)$$

Common factor of the above quadratic expressions is $(2x + 1)$.

Therefore, GCD $= 2(2x + 1)$.

3.5.2 Greatest common divisor of polynomials using division algorithm

First let us consider the simple case of finding GCD of 924 and 105.

$924 = 8 \times 105 + 84$

$105 = 1 \times 84 + 21,$ (or)

$84 = 4 \times 21 + 0,$

21 is the GCD of 924 and 105

$$\begin{array}{r|l} & \ \ 8 \\ \hline 105 & 924 \\ & 840 \\ \hline & \ \ 84 \end{array} \quad \begin{array}{r|l} & \ \ 1 \\ \hline 84 & 105 \\ & \ \ 84 \\ \hline & \ \ 21 \end{array} \quad \begin{array}{r|l} & \ \ 4 \\ \hline 21 & 84 \\ & 84 \\ \hline & \ \ 0 \end{array}$$

Similar technique works with polynomials when they have GCD.

3.5.3 Least Common Multiple (LCM)

The least common multiple of two or more algebraic expressions is the expression of lowest degree which is divisible by each of them without remainder. For example, consider the simple expressions a^4, a^3, a^6.

Now, a^6, a^7, a^8, $\cdots$ are common multiples of a^3, a^4 and a^6.

Of all the common multiples, the least common multiple is a^6

Hence LCM of a^4, a^3, a^6 is a^6. Similarly, a^3b^7 is the LCM of a^3b^4, ab^5, a^2b^7.

We shall consider some more examples of finding LCM.

Example 3.22

Find the LCM of the following.

(i) 90, 150, 225 (ii) $35a^2c^3b$, $42a^3cb^2$, $30ac^2b^3$

(iii) $(a-1)^5(a+3)^2, (a-2)^2(a-1)^3(a+3)^4$

(iv) $x^3+y^3, x^3-y^3, x^4+x^2y^2+y^4$

Solution

(i) Now,
$$90 = 2\times3\times3\times5 = 2^1\times3^2\times5^1$$
$$150 = 2\times3\times5\times5 = 2^1\times3^1\times5^2$$
$$225 = 3\times3\times5\times5 = 3^2\times5^2$$

The product $2^1\times3^2\times5^2 = 450$ is the required LCM.

(ii) Now, LCM of 35, 42 and 30 is $5\times7\times6 = 210$

Hence, the required LCM $= 210\times a^3\times c^3\times b^3 = 210a^3c^3b^3$.

(iii) Now, LCM of $(a-1)^5(a+3)^2, (a-2)^2(a-1)^3(a+3)^4$ is $(a-1)^5(a+3)^4(a-2)^2$.

(iv) Let us first find the factors for each of the given expressions.

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$
$$x^3-y^3 = (x-y)(x^2+xy+y^2)$$
$$x^4+x^2y^2+y^4 = (x^2+y^2)^2-x^2y^2 = (x^2+xy+y^2)(x^2-xy+y^2)$$

Thus,
$$\text{LCM} = (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$$
$$= (x^3+y^3)(x^3-y^3) = x^6-y^6.$$

Exercise 3.7

Find the LCM of the following.

1. x^3y^2, xyz
2. $3x^2yz$, $4x^3y^3$
3. a^2bc, b^2ca, c^2ab
4. $66a^4b^2c^3$, $44a^3b^4c^2$, $24a^2b^3c^4$
5. a^{m+1}, a^{m+2}, a^{m+3}
6. x^2y+xy^2, x^2+xy
7. $3(a-1)$, $2(a-1)^2$, (a^2-1)
8. $2x^2-18y^2$, $5x^2y+15xy^2$, x^3+27y^3
9. $(x+4)^2(x-3)^3$, $(x-1)(x+4)(x-3)^2$
10. $10(9x^2+6xy+y^2)$, $12(3x^2-5xy-2y^2)$, $14(6x^4+2x^3)$.

3.5.4 Relation between LCM and GCD

We know that the product of two positive integers is equal to the product of their LCM and GCD. For example, $21 \times 35 = 105 \times 7$, where LCM (21,35) =105 and GCD (21,35) = 7.

In the same way, we have the following result:

The product of any two polynomials is equal to the product of their LCM and GCD.

That is, $f(x) \times g(x) = \text{LCM}\,(f(x), g(x)) \times \text{GCD}\,(f(x), g(x))$.

Let us justify this result with an example.

Let $f(x) = 12(x^4 - x^3)$ and $g(x) = 8(x^4 - 3x^3 + 2x^2)$ be two polynomials.

Now, $f(x) = 12(x^4 - x^3) = 2^2 \times 3 \times x^3 \times (x-1)$ (1)

Also, $g(x) = 8(x^4 - 3x^3 + 2x^2) = 2^3 \times x^2 \times (x-1) \times (x-2)$ (2)

From (1) and (2) we get,

$\text{LCM}\,(f(x), g(x)) = 2^3 \times 3^1 \times x^3 \times (x-1) \times (x-2) = 24x^3(x-1)(x-2)$

$\text{GCD}\,(f(x), g(x)) = 4x^2(x-1)$

Therefore, $\text{LCM} \times \text{GCD} = 24x^3(x-1)(x-2) \times 4x^2(x-1)$

$= 96x^5(x-1)^2(x-2)$ (3)

Also, $f(x) \times g(x) = 12x^3(x-1) \times 8x^2(x-1)(x-2)$

$= 96x^5(x-1)^2(x-2)$ (4)

From (3) and (4) we obtain, $\text{LCM} \times \text{GCD} = f(x) \times g(x)$.

Thus, the product of LCM and GCD of two polynomials is equal to the product of the two polynomials. Further, if $f(x)$, $g(x)$ and one of LCM and GCD are given, then the other can be found without ambiguity because LCM and GCD are unique, except for a factor of –1.

Example 3.23

The GCD of $x^4 + 3x^3 + 5x^2 + 26x + 56$ and $x^4 + 2x^3 - 4x^2 - x + 28$ is $x^2 + 5x + 7$. Find their LCM.

Solution Let $f(x) = x^4 + 3x^3 + 5x^2 + 26x + 56$ and $g(x) = x^4 + 2x^3 - 4x^2 - x + 28$

Given that $\text{GCD} = x^2 + 5x + 7$. Also, we have $\text{GCD} \times \text{LCM} = f(x) \times g(x)$.

Thus, $\text{LCM} = \dfrac{f(x) \times g(x)}{\text{GCD}}$ (1)

Now, GCD divides both $f(x)$ and $g(x)$.

Let us divide $f(x)$ by the GCD.

$$\begin{array}{r|rrrrr}
 & 1 & -2 & 8 & & \\
\hline
1\ \ 5\ \ 7 & 1 & 3 & 5 & 26 & 56 \\
 & 1 & 5 & 7 & & \\
\hline
 & & -2 & -2 & 26 & \\
 & & -2 & -10 & -14 & \\
\hline
 & & & 8 & 40 & 56 \\
 & & & 8 & 40 & 56 \\
\hline
 & & & & 0 &
\end{array}$$

When $f(x)$ is divided by GCD, we get the quotient as $x^2 - 2x + 8$.

Now, (1) $\Longrightarrow$ LCM $= (x^2 - 2x + 8) \times g(x)$

Thus, LCM $= (x^2 - 2x + 8)(x^4 + 2x^3 - 4x^2 - x + 28)$.

Note

In the above problem, we can also divide $g(x)$ by GCD and multiply the quotient by $f(x)$ to get the required LCM.

Example 3.24

The GCD and LCM of two polynomials are $x + 1$ and $x^6 - 1$ respectively. If one of the polynomials is $x^3 + 1$, find the other.

Solution Given GCD $= x + 1$ and LCM $= x^6 - 1$

Let $f(x) = x^3 + 1$.

We know that LCM $\times$ GCD $= f(x) \times g(x)$

$$\Longrightarrow g(x) = \frac{\text{LCM} \times \text{GCD}}{f(x)} = \frac{(x^6 - 1)(x + 1)}{x^3 + 1}$$

$$= \frac{(x^3 + 1)(x^3 - 1)(x + 1)}{x^3 + 1} = (x^3 - 1)(x + 1)$$

Hence, $g(x) = (x^3 - 1)(x + 1)$.

Example 3.25

Simplify the rational expressions into lowest forms.

(i) $\frac{5x+20}{7x+28}$ (ii) $\frac{x^3-5x^2}{3x^3+2x^4}$

(iii) $\frac{6x^2-5x+1}{9x^2+12x-5}$ (iv) $\frac{(x-3)(x^2-5x+4)}{(x-1)(x^2-2x-3)}$

Solution

(i) Now, $\frac{5x+20}{7x+28} = \frac{5(x+4)}{7(x+4)} = \frac{5}{7}$

(ii) Now, $\frac{x^3-5x^2}{3x^3+2x^4} = \frac{x^2(x-5)}{x^3(2x+3)} = \frac{x-5}{x(2x+3)}$

(iii) Let $p(x) = 6x^2-5x+1 = (2x-1)(3x-1)$ and

$q(x) = 9x^2+12x-5 = (3x+5)(3x-1)$

Therefore, $\frac{p(x)}{q(x)} = \frac{(2x-1)(3x-1)}{(3x+5)(3x-1)} = \frac{2x-1}{3x+5}$

(iv) Let $f(x) = (x-3)(x^2-5x+4) = (x-3)(x-1)(x-4)$ and

$g(x) = (x-1)(x^2-2x-3) = (x-1)(x-3)(x+1)$

Therefore, $\frac{f(x)}{g(x)} = \frac{(x-3)(x-1)(x-4)}{(x-1)(x-3)(x+1)} = \frac{x-4}{x+1}$

Example 3.28

Simplify (i) $\frac{x+2}{x+3}+\frac{x-1}{x-2}$ (ii) $\frac{x+1}{(x-1)^2}+\frac{1}{x+1}$ (iii) $\frac{x^2-x-6}{x^2-9}+\frac{x^2+2x-24}{x^2-x-12}$

Solution

(i) $\frac{x+2}{x+3}+\frac{x-1}{x-2} = \frac{(x+2)(x-2)+(x-1)(x+3)}{(x+3)(x-2)} = \frac{2x^2+2x-7}{x^2+x-6}$

(ii) $\frac{x+1}{(x-1)^2}+\frac{1}{x+1} = \frac{(x+1)^2+(x-1)^2}{(x-1)^2(x+1)} = \frac{2x^2+2}{(x-1)^2(x+1)}$

$= \frac{2x^2+2}{x^3-x^2-x+1}$

(iii) $\frac{x^2-x-6}{x^2-9}+\frac{x^2+2x-24}{x^2-x-12} = \frac{(x-3)(x+2)}{(x+3)(x-3)}+\frac{(x+6)(x-4)}{(x+3)(x-4)}$

$= \frac{x+2}{x+3}+\frac{x+6}{x+3} = \frac{x+2+x+6}{x+3} = \frac{2x+8}{x+3}$

3.7.1 Square root by factorization method

Example 3.31

Find the square root of

(i) $121(x-a)^4(x-b)^6(x-c)^{12}$ (ii) $\dfrac{81x^4y^6z^8}{64w^{12}s^{14}}$ (iii) $(2x+3y)^2-24xy$

Solution

(i) $\sqrt{121(x-a)^4(x-b)^6(x-c)^{12}} = 11|(x-a)^2(x-b)^3(x-c)^6|$

(ii) $\sqrt{\dfrac{81x^4y^6z^8}{64w^{12}s^{14}}} = \dfrac{9}{8}\left|\dfrac{x^2y^3z^4}{w^6s^7}\right|$

(iii) $\sqrt{(2x+3y)^2-24xy} = \sqrt{4x^2+12xy+9y^2-24xy} = \sqrt{(2x-3y)^2}$

Example 3.42

The sum of a number and its reciprocal is $5\frac{1}{5}$. Find the number.

Solution Let x denote the required number. Then its reciprocal is $\frac{1}{x}$

By the given condition, $x+\dfrac{1}{x} = 5\dfrac{1}{5} \implies \dfrac{x^2+1}{x} = \dfrac{26}{5}$

So, $5x^2-26x+5 = 0$

$\implies 5x^2-25x-x+5 = 0$

That is, $(5x-1)(x-5) = 0 \implies x = 5$ or $\dfrac{1}{5}$

Thus, the required numbers are $5, \dfrac{1}{5}$.

Example 3.43

The base of a triangle is 4cm longer than its altitude. If the area of the triangle is 48 sq. cm, then find its base and altitude.

Solution Let the altitude of the triangle be x cm.

By the given condition, the base of the triangle is $(x+4)$ cm.

Now, the area of the triangle = $\frac{1}{2}$(base) × height

By the given condition $\frac{1}{2}(x+4)(x) = 48$

$\implies x^2+4x-96 = 0 \implies (x+12)(x-8) = 0$

$\implies x = -12$ or 8

But $x = -12$ is not possible (since the length should be positive)

Therefore, $x = 8$ and hence, $x+4 = 12$.

Thus, the altitude of the triangle is 8cm and the base of the triangle is 12cm.

Example 3.44

A car left 30 minutes later than the scheduled time. In order to reach its destination 150km away in time, it has to increase its speed by 25km/hr from its usual speed. Find its usual speed.

Solution Let the usual speed of the car be x km/hr.

Thus, the increased speed of the car is $(x+25)$ km/hr

Total distance $= 150$ km; Time taken $= \frac{\text{Distance}}{\text{Speed}}$.

Let T_1 and T_2 be the time taken in hours by the car to cover the given distance in scheduled time and decreased time (as the speed is increased) respectively.

By the given information $T_1 - T_2 = \frac{1}{2}$hr (30 minutes $= \frac{1}{2}$hr)

$$\Longrightarrow \frac{150}{x} - \frac{150}{x+25} = \frac{1}{2} \Longrightarrow 150\left[\frac{x+25-x}{x(x+25)}\right] = \frac{1}{2}$$

$$\Longrightarrow x^2 + 25x - 7500 = 0 \Longrightarrow (x+100)(x-75) = 0$$

Thus, $x = 75$ or -100, but $x = -100$ is not an admissible value.

Therefore, the usual speed of the car is 75 km/hr.

Note

If α and β are the roots of $ax^2 + bx + c = 0$, then many expressions in α and β like $\alpha^2 + \beta^2$, $\alpha^2\beta^2$, $\alpha^2 - \beta^2$ etc., can be evaluated using the values of $\alpha + \beta$ and $\alpha\beta$.

Let us write some results involving α and β.

(i) $|\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

(ii) $\alpha^2 + \beta^2 = [(\alpha+\beta)^2 - 2\alpha\beta]$

(iii) $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta) = (\alpha+\beta)[\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}]$ only if $\alpha \geq \beta$

(iv) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

(v) $\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$

(vi) $\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2 = [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

(vii) $\alpha^4 - \beta^4 = (\alpha+\beta)(\alpha-\beta)(\alpha^2+\beta^2)$

8.2.1 Right Circular Cylinder

If we take a number of circular sheets of paper or cardboard of the same shape and size and stack them up in a vertical pile, then by this process, we shall obtain a solid object known as a Right Circular Cylinder. Note that it has been kept at right angles to the base, and the base is circular. (See Fig. 8.3)

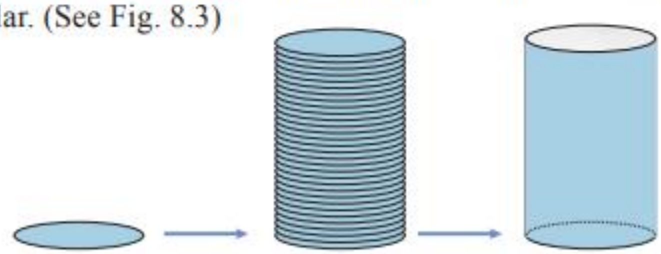


Fig. 8.3

Definition

If a rectangle revolves about its one side and completes a full rotation, the solid thus formed is called a right circular cylinder.

Activity

Let $ABCD$ be a rectangle. Assume that it revolves about its side AB and completes a full rotation. This revolution generates a right circular cylinder as shown in the figures. AB is called the axis of the cylinder. The length AB is the length or the height of the cylinder and AD or BC is called its radius.

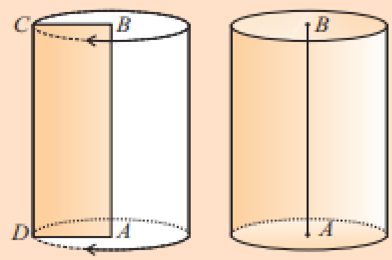


Fig. 8.4

Note

(i) If the base of a cylinder is not circular then it is called oblique cylinder.

(ii) If the base is circular but not perpendicular to the axis of the cylinder, then the cylinder is called circular cylinder.

(iii) If the axis is perpendicular to the circular base, then the cylinder is called right circular cylinder.

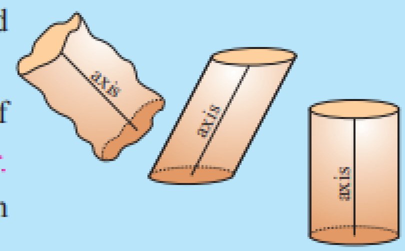


Fig. 8.5

(i) Curved Surface area of a solid right circular cylinder

In the adjoining figure, the bottom and top face of the right circular cylinder are concurrent circular regions, parallel to each other. The vertical surface of the cylinder is curved and hence its area is called the curved surface or lateral surface area of the cylinder.

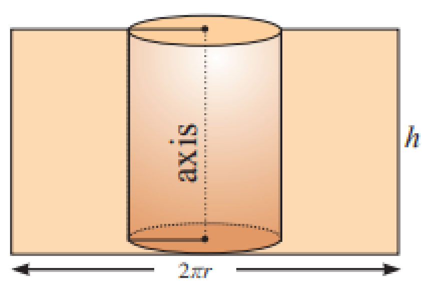


Fig. 8.6

Curved Surface Area of a cylinder, CSA = Circumference of the base $\times$ Height $= 2\pi r \times h$

$= 2\pi rh$ sq. units.

(ii) Total Surface Area of a solid right circular cylinder

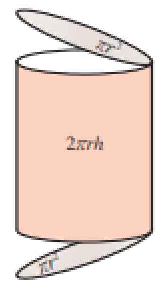


Fig. 8.7

Total Surface Area, TSA = Area of the Curved Surface Area + 2 × Base Area

$$= 2\pi rh + 2\times \pi r^2$$

Thus, TSA $= 2\pi r(h + r)$ sq.units.

(iii) Right circular hollow cylinder

Solids like iron pipe, rubber tube, etc., are in the shape of hollow cylinders. For a hollow cylinder of height h with external and internal radii R and r respectively,

we have, curved surface area, CSA = External surface area + Internal surface area

$$= 2\pi Rh + 2\pi rh$$

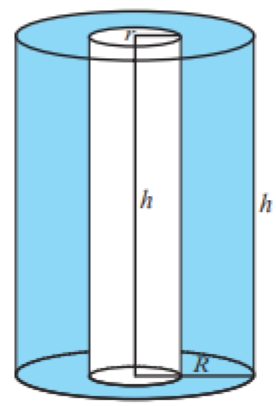
Thus, CSA $= 2\pi h(R + r)$ sq.units

Total surface area, TSA = CSA + 2 × Base area

$$= 2\pi h(R + r) + 2\times[\pi R^2 - \pi r^2]$$

$$= 2\pi h(R + r) + 2\pi(R + r)(R - r)$$

$\therefore$ TSA $= 2\pi(R + r)(R - r + h)$ sq.units.



Remark

Thickness of the hollow cylinder, w = $R - r$.

Fig. 8.8

Note

In this chapter, for π we take an approximate value $\frac{22}{7}$ whenever it is required.

Example 8.1

A solid right circular cylinder has radius 7cm and height 20cm. Find its (i) curved surface area and (ii) total surface area. (Take $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the solid right circular cylinder respectively.

Given that $r = 7$cm and $h = 20$cm

Curved surface area, CSA $= 2\pi rh$

$= 2 \times \frac{22}{7} \times 7 \times 20$

Thus, the curved surface area = 880 sq.cm

Now, the total surface area $= 2\pi r(h + r)$

$= 2 \times \frac{22}{7} \times 7 \times [20 + 7] = 44 \times 27$

Thus, the total surface area = 1188 sq.cm.

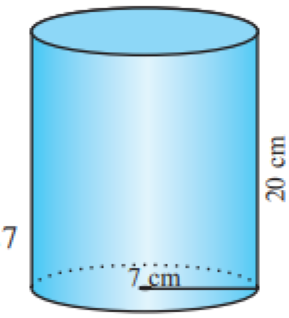


Fig. 8.9

Example 8.2

If the total surface area of a solid right circular cylinder is 880 sq.cm and its radius is 10 cm, find its curved surface area. (Take $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the solid right circular cylinder respectively.

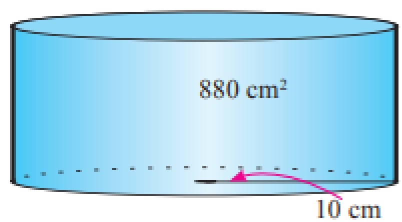


Fig. 8.10

Let S be the total surface area of the solid right circular cylinder.

Given that $r = 10$ cm and $S = 880$ cm^2

Now, $S = 880 \implies 2\pi r[h + r] = 880$

$\implies 2 \times \frac{22}{7} \times 10[h + 10] = 880$

$\implies h + 10 = \frac{880 \times 7}{2 \times 22 \times 10}$

$\implies h + 10 = 14$

Thus, the height of the cylinder, $h = 4$cm

Now, the curved surface area, CSA is

$2\pi rh = 2 \times \frac{22}{7} \times 10 \times 4 = \frac{1760}{7}$

Thus, the curved surface area of the cylinder $= 251\frac{3}{7}$ sq.cm.

Aliter :

CSA = TSA – 2× Area of the base

$= 880 - 2 \times \pi r^2$

$= 880 - 2 \times \frac{22}{7} \times 10^2$

$= \frac{1760}{7} = 251\frac{3}{7}$ sq.cm.

Example 8.3

The ratio between the base radius and the height of a solid right circular cylinder is 2 : 5. If its curved surface area is $\frac{3960}{7}$ sq.cm, find the height and radius. (use $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the right circular cylinder respectively.

Given that $r : h = 2 : 5 \implies \frac{r}{h} = \frac{2}{5}$. Thus, $r = \frac{2}{5}h$

Now, the curved surface area, CSA $= 2\pi rh$

$$\implies \quad 2 \times \frac{22}{7} \times \frac{2}{5} \times h \times h = \frac{3960}{7}$$

$$\implies \quad h^2 = \frac{3960 \times 7 \times 5}{2 \times 22 \times 2 \times 7} = 225$$

Thus, $h = 15 \implies r = \frac{2}{5}h = 6.$

Hence, the height of the cylinder is 15 cm and the radius is 6 cm.

Example 8.5

The internal and external radii of a hollow cylinder are 12 cm and 18 cm respectively. If its height is 14cm, then find its curved surface area and total surface area. (Take $\pi = \frac{22}{7}$)

Solution Let r, R and h be the internal and external radii and the height of a hollow cylinder respectively.

Given that $r = 12$ cm, $R = 18$ cm, $h = 14$ cm

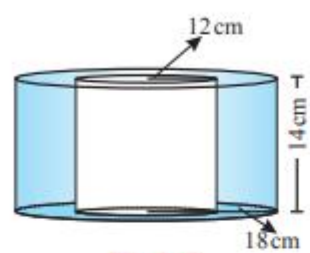


Fig. 8.12

Now, curved surface area, CSA $= 2\pi h(R+r)$

Thus, CSA $= 2 \times \frac{22}{7} \times 14 \times (18 + 12)$

$= 2640$ sq.cm

Total surface area, TSA $= 2\pi(R + r)(R - r + h)$

$$= 2 \times \frac{22}{7} \times (18 + 12)(18 - 12 + 14)$$

$$= 2 \times \frac{22}{7} \times 30 \times 20 = \frac{26400}{7}.$$

Thus, the total surface area $= 3771\frac{3}{7}$ sq.cm.

Fig. 8.13

The length AB is called the height of the cone.

The length BC is called the radius of its base ($BC = r$). The length AC is called the slant height l of the cone ($AC = AD = l$).

In the right angled $\triangle ABC$

We have, $l = \sqrt{h^2 + r^2}$ (Pythagoras theorem)

$$h = \sqrt{l^2 - r^2}$$

$$r = \sqrt{l^2 - h^2}$$

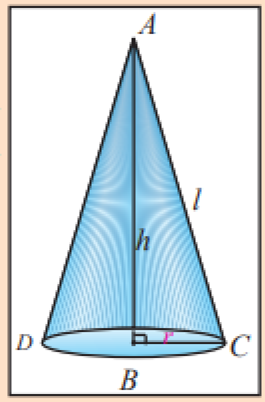


Fig. 8.14

(i) Curved surface area of a hollow cone

Let us consider a sector with radius l and central angle θ°. Let L denote the length of the arc. Thus, $\dfrac{2\pi l}{L} = \dfrac{360^\circ}{\theta^\circ}$

$$\Longrightarrow \quad L = 2\pi l \times \frac{\theta^\circ}{360^\circ} \qquad (1)$$

Now, join the radii of the sector to obtain a right circular cone.

Let r be the radius of the cone.

Hence, $L = 2\pi r$

From (1) we obtain,

$$2\pi r = 2\pi l \times \frac{\theta^\circ}{360^\circ}$$

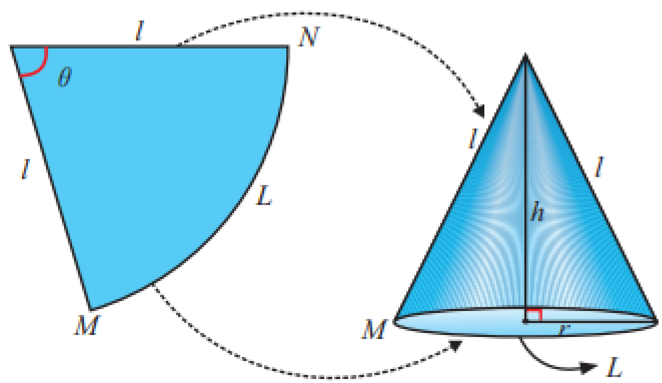


Fig. 8.16

$$\Longrightarrow \quad r = l\left(\frac{\theta^\circ}{360^\circ}\right)$$

$$\Longrightarrow \quad \frac{r}{l} = \left(\frac{\theta^\circ}{360^\circ}\right)$$

Let A be the area of the sector. Then

$$\frac{\pi l^2}{A} = \frac{360^\circ}{\theta^\circ} \qquad (2)$$

Remarks

When a sector of a circle is folded into a cone, the following conversions are taking place:

Sector	Cone
Radius (l) →	Slant height (l)
Arc Length (L) →	Perimeter of the base $2\pi r$
Area →	Curved Surface Area πrl

Then the curved surface area of the cone $\}$ = Area of the sector

Thus, the area of the curved surface of the cone $\}$ $A = \pi l^2\left(\frac{\theta^\circ}{360^\circ}\right) = \pi l^2\left(\frac{r}{l}\right)$.

Hence, the curved surface area of the cone = πrl sq.units.

(ii) Total surface area of the solid right circular cone

Total surface area of the solid cone = $\left\{\begin{array}{l}\text{Curved surface area of the cone} \\ + \text{ Area of the base}\end{array}\right.$

$$= \pi rl + \pi r^2$$

Total surface area of the solid cone = $\pi r(l + r)$ sq.units.

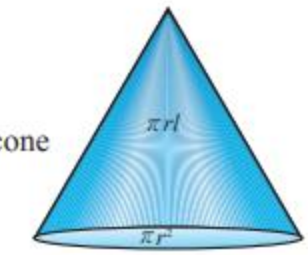


Fig. 8.17

Example 8.6

Radius and slant height of a solid right circular cone are 35cm and 37cm respectively. Find the curved surface area and total surface area of the cone. (Take $\pi = \frac{22}{7}$)

Solution Let r and l be the radius and the slant height of the solid right circular c respectively.

$$r = 35 \text{ cm}, \ l = 37 \text{ cm}$$

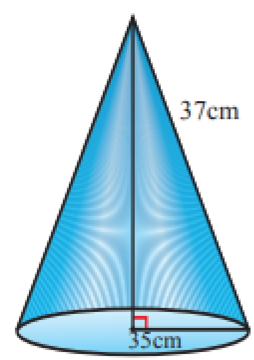
Curved surface area, CSA = $\pi rl = \pi(35)(37)$

CSA = 4070 sq.cm

Total surface area, TSA = $\pi r[l + r]$

$$= \frac{22}{7} \times 35 \times [37 + 35]$$

Thus, TSA = 7920 sq.cm.



Example 8.7

Let O and C be the centre of the base and the vertex of a right circular cone. Let B be any point on the circumference of the base. If the radius of the cone is 6 cm and if $\angle OBC = 60^\circ$, then find the height and curved surface area of the cone.

Solution Given that radius OB = 6 cm and $\angle OBC = 60^\circ$.

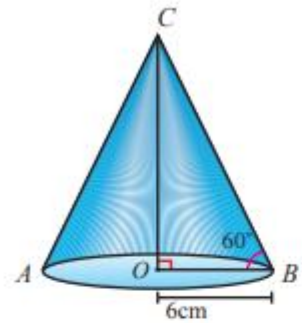


Fig. 8.19

In the right angled $\triangle OBC$,

$$\cos 60^\circ = \frac{OB}{BC}$$

$$\Longrightarrow \quad BC = \frac{OB}{\cos 60^\circ}$$

$$\therefore \quad BC = \frac{6}{(\frac{1}{2})} = 12 \text{ cm}$$

Thus, the slant height of the cone, $l = 12$ cm

In the right angled $\triangle OBC$, we have

$$\tan 60^\circ = \frac{OC}{OB}$$

$$\Longrightarrow \quad OC = OB \tan 60^\circ = 6\sqrt{3}$$

Thus, the height of the cone, OC = $6\sqrt{3}$ cm

Now, the curved surface area is $\pi rl = \pi \times 6 \times 12 = 72\pi \text{ cm}^2$.

Example 8.8

A sector containing an angle of 120° is cut off from a circle of radius 21 cm and folded into a cone. Find the curved surface area of the cone. (Take $\pi = \frac{22}{7}$)

Solution Let r be the base radius of the cone.

Angle of the sector, $\theta = 120^\circ$

Radius of the sector, R = 21 cm

When the sector is folded into a right circular cone, we have

circumference of the base of the cone

= Length of the arc

$$\implies 2\pi r = \frac{\theta}{360^\circ} \times 2\pi R$$

$$\implies r = \frac{\theta}{360^\circ} \times R$$

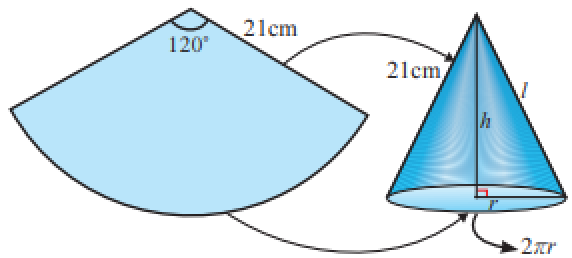


Fig. 8.20

Thus, the base radius of the cone, $r = \frac{120^\circ}{360^\circ} \times 21 = 7$ cm.

Also, the slant height of the cone ,

l = Radius of the sector

Thus, $l = R \implies l = 21$ cm.

Now , the curved surface area of the cone,

$$\text{CSA} = \pi r l$$

$$= \frac{22}{7} \times 7 \times 21 = 462.$$

Thus, the curved surface area of the cone is 462 sq.cm.

Aliter :

CSA of the cone = Area of the sector

$$= \frac{\theta^\circ}{360^\circ} \times \pi \times R^2$$

$$= \frac{120}{360} \times \frac{22}{7} \times 21 \times 21$$

$$= 462 \text{ sq.cm.}$$

8.2.3 Sphere

If a circular disc is rotated about one of its diameter, the solid thus generated is called sphere. Thus sphere is a 3- dimensional object which has surface area and volume.

(i) Curved surface area of a solid sphere

Activity

Take a circular disc, paste a string along a diameter of the disc and rotate it 360°. The object so created looks like a ball. The new solid is called sphere.

The following activity may help us to visualise the surface area of a sphere as four times the area of the circle with the same radius.

- Take a plastic ball.
- Fix a pin at the top of the ball.
- Wind a uniform thread over the ball so as to cover the whole curved surface area.
- Unwind the thread and measure the length of the thread used.
- Cut the thread into four equal parts.
- Place the strings as shown in the figures.

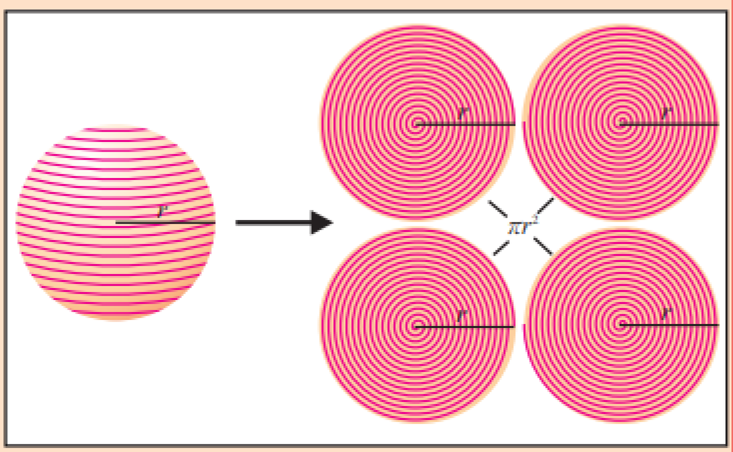


Fig. 8.21

- Measure the radius of the sphere and the circles formed.

Now, the radius of the sphere = radius of the four equal circles.

Thus, curved surface area of the sphere, CSA = $4 \times$ Area of the circle = $4 \times \pi r^2$

$\therefore$ The curved surface area of a sphere = $4\pi r^2$ sq. units.

(ii) Solid hemisphere

A plane passing through the centre of a solid sphere divides the sphere into two equal parts. Each part of the sphere is called a solid hemisphere.

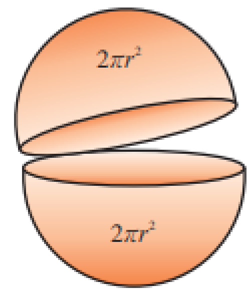


Fig. 8.22

$$\text{Curved surface area of a hemisphere} = \frac{\text{CSA of the Sphere}}{2}$$

$$= \frac{4\pi r^2}{2} = 2\pi r^2 \text{ sq.units.}$$

Total surface area of a hemisphere, TSA = Curved Surface Area + Area of the base Circle

$$= 2\pi r^2 + \pi r^2$$

$$= 3\pi r^2 \text{ sq.units.}$$

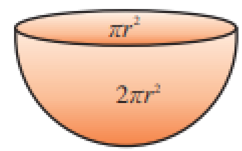


Fig. 8.23

(iii) Hollow hemisphere

Let R and r be the outer and inner radii of the hollow hemisphere.

Now, its curved surface area = Outer surface area + Inner surface area

$$= 2\pi R^2 + 2\pi r^2$$

$$= 2\pi(R^2 + r^2) \text{ sq.units.}$$

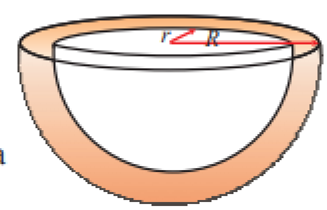


Fig. 8.24

$$\text{The total surface area} = \begin{cases} \text{Outer surface area + Inner surface area} \\ + \text{ Area at the base} \end{cases}$$

$$= 2\pi R^2 + 2\pi r^2 + \pi(R^2 - r^2)$$

$$= 2\pi(R^2 + r^2) + \pi(R + r)(R - r) \text{ sq.units.}$$

$$= \pi(3R^2 + r^2) \text{ sq. units}$$

Example 8.9

A hollow sphere in which a circus motorcyclist performs his stunts, has an inner diameter of 7 m. Find the area available to the motorcyclist for riding. (Take $\pi = \frac{22}{7}$)

Solution Inner diameter of the hollow sphere, $2r = 7$ m.

Available area to the motorcyclist for riding $=$ Inner surface area of the sphere

$$= 4\pi r^2 = \pi(2r)^2$$

$$= \frac{22}{7} \times 7^2$$

Available area to the motorcyclist for riding $= 154$ sq.m.

Example 8.10

Total surface area of a solid hemisphere is 675π sq.cm. Find the curved surface area of the solid hemisphere.

Solution Given that the total surface area of the solid hemisphere,

$$3\pi r^2 = 675\pi \text{ sq. cm}$$

$$\Longrightarrow \quad r^2 = 225$$

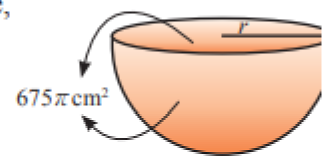


Fig. 8.25

Now, the curved surface area of the solid hemisphere,

$$\text{CSA} = 2\pi r^2 = 2\pi \times 225 = 450\pi \text{ sq.cm.}$$

Example 8.11

The thickness of a hemispherical bowl is 0.25 cm. The inner radius of the bowl is 5cm. Find the outer curved surface area of the bowl.(Take $\pi = \frac{22}{7}$)

Solution Let r, R and w be the inner and outer radii and thickness of the hemispherical bowl respectively.

Given that $r = 5$ cm, $w = 0.25$ cm

$\therefore \quad R = r + w = 5 + 0.25 = 5.25$ cm

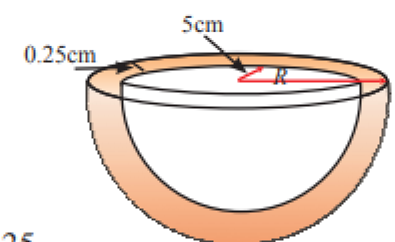


Fig. 8.26

Now, outer surface area of the bowl $= 2\pi R^2$

$$= 2 \times \frac{22}{7} \times 5.25 \times 5.25$$

Thus, the outer surface area of the bowl $= 173.25$ sq.cm.

8.3 Volume

So far we have seen the problems related to the surface area of some solids. Now we shall learn how to calculate volumes of some familiar solids. Volume is literally the 'amount of space filled'. The volume of a solid is a numerical characteristic of the solid.

For example, if a body can be decomposed into finite set of unit cubes (cubes of unit sides), then the volume is equal to the number of these cubes.

The cube in the figure, has a volume

$= \text{length} \times \text{width} \times \text{height}$

$= 1 \text{ cm} \times 1 \text{ cm} \times 1 \text{ cm} = 1 \text{ cm}^3$.

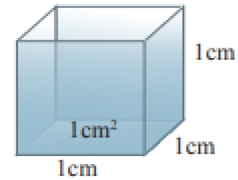


Fig. 8.27

If we say that the volume of an object is 100 cu.cm, then it implies that we need 100 cubes each of 1 cm^3 volume to fill this object completely.

Just like surface area, volume is a positive quantity and is invariant with respect to displacement. Volumes of some solids are illustrated below.

8.3.1 Volume of a right circular cylinder

(i) Volume of a solid right circular cylinder

The volume of a solid right circular cylinder is the product of the base area and height.

That is, the volume of the cylinder, V = Area of the base $\times$ height

$= \pi r^2 \times h$

Thus, the volume of a cylinder, $V = \pi r^2 h$ cu. units.

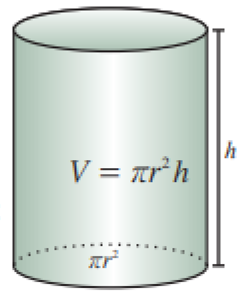


Fig. 8.28

(ii) Volume of a hollow cylinder (Volume of the material used)

Let R and r be the external and internal radii of a hollow right circular cylinder respectively. Let h be its height.

Then, the volume, $V = \begin{Bmatrix}\text{Volume of the}\\ \text{outer cylinder}\end{Bmatrix} - \begin{Bmatrix}\text{Volume of the}\\ \text{inner cylinder}\end{Bmatrix}$

$= \pi R^2 h - \pi r^2 h$

Hence, the volume of a hollow cylinder,

$V = \pi h (R^2 - r^2)$ cu. units.

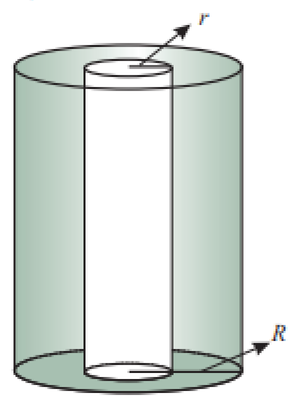


Fig. 8.29

Example 8.12

If the curved surface area of a right circular cylinder is 704 sq.cm, and height is 8 cm, find the volume of the cylinder in litres. (Take $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the right circular cylinder respectively.

Given that $h = 8$ cm and CSA = 704 sq.cm

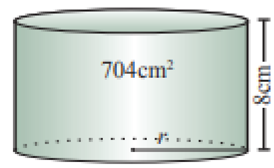


Fig. 8.30

Now, CSA = 704

$\Longrightarrow \quad 2\pi rh = 704$

$2 \times \frac{22}{7} \times r \times 8 = 704$

$\therefore \quad r = \frac{704 \times 7}{2 \times 22 \times 8} = 14$ cm

Thus, the volume of the cylinder, $V = \pi r^2 h$

$= \frac{22}{7} \times 14 \times 14 \times 8$

$= 4928$ cu.cm.

Hence, the volume of the cylinder = 4.928 litres. (1000 cu.cm = 1 litre)

Example 8.13

A hollow cylindrical iron pipe is of length 28 cm. Its outer and inner diameters are 8 cm and 6 cm respectively. Find the volume of the pipe and weight of the pipe if 1 cu.cm of iron weighs 7 gm.(Take $\pi = \frac{22}{7}$)

Solution Let r, R and h be the inner, outer radii and height of the hollow cylindrical pipe respectively.

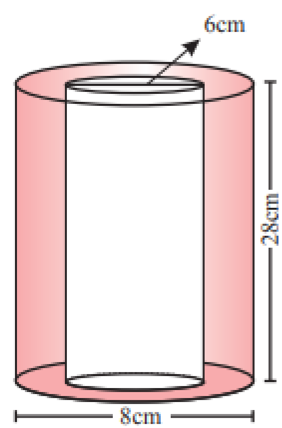


Fig. 8.31

Given that $2r = 6$ cm, $2R = 8$ cm, $h = 28$ cm

Now, the volume of the pipe, $V = \pi \times h \times (R + r)(R - r)$

$= \frac{22}{7} \times 28 \times (4 + 3)(4 - 3)$

$\therefore$ Volume, $V = 616$ cu. cm

Weight of 1 cu.cm of the metal = 7 gm

Weight of the 616 cu.cm of metal = 7×616 gm

Thus, the weight of the pipe = 4.312 kg.

Example 8.14

Base area and volume of a solid right circular cylinder are 13.86 sq.cm, and 69.3 cu.cm respectively. Find its height and curved surface area.(Take $\pi = \frac{22}{7}$)

Solution Let A and V be the base area and volume of the solid right circular cylinder respectively.

Given that the base area, $A = \pi r^2 = 13.86$ sq.cm and

volume, $V = \pi r^2 h = 69.3$ cu.cm.

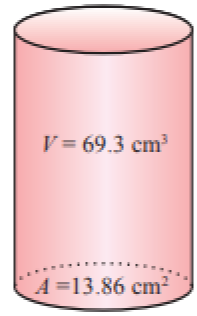


Fig. 8.32

$$\text{Thus,} \quad \pi r^2 h = 69.3$$
$$\Longrightarrow \quad 13.86 \times h = 69.3$$
$$\therefore \quad h = \frac{69.3}{13.86} = 5 \text{ cm.}$$

Now, the base area $= \pi r^2 = 13.86$

$$\frac{22}{7} \times r^2 = 13.86$$
$$r^2 = 13.86 \times \frac{7}{22} = 4.41 \quad \Longrightarrow \quad r = \sqrt{4.41} = 2.1 \text{ cm.}$$

Now, Curved surface area, $\text{CSA} = 2\pi rh$

$$= 2 \times \frac{22}{7} \times 2.1 \times 5$$

Thus, $\text{CSA} = 66$ sq.cm.

Example 8.15

The volume of a solid right circular cone is 4928 cu. cm. If its height is 24 cm, then find the radius of the cone. (Take $\pi = \frac{22}{7}$)

Solution Let r, h and V be the radius, height and volume of a solid cone respectively.

Given that $V = 4928$ cu.cm and $h = 24$ cm

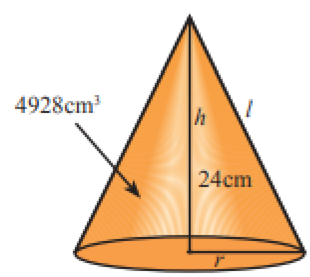


Fig. 8.34

$$\text{Thus, we have} \quad \frac{1}{3}\pi r^2 h = 4928$$
$$\Longrightarrow \quad \frac{1}{3} \times \frac{22}{7} \times r^2 \times 24 = 4928$$
$$\Longrightarrow \quad r^2 = \frac{4928 \times 3 \times 7}{22 \times 24} = 196.$$

Thus, the base radius of the cone, $r = \sqrt{196} = 14$ cm.

Note

* Curved surface area of a frustum of a cone $= \pi(R + r)l$, where $l = \sqrt{h^2 + (R - r)^2}$

* Total surface area of a frustum of a the cone $= \pi l(R + r) + \pi R^2 + \pi r^2, l = \sqrt{h^2 + (R - r)^2}$

(* Not to be used for examination purpose)

Example 8.16

The radii of two circular ends of a frustum shaped bucket are 15 cm and 8 cm. If its depth is 63 cm, find the capacity of the bucket in litres. (Take $\pi = \frac{22}{7}$)

Solution Let R and r are the radii of the circular ends at the top and bottom and h be the depth of the bucket respectively.

Given that $R = 15$ cm , $r = 8$ cm and $h = 63$ cm.

The volume of the bucket (frustum)

$$= \frac{1}{3}\pi h(R^2 + r^2 + Rr)$$

$$= \frac{1}{3} \times \frac{22}{7} \times 63 \times (15^2 + 8^2 + 15 \times 8)$$

$$= 26994 \text{ cu.cm.}$$

$$= \frac{26994}{1000} \text{ litres} \qquad (1000 \text{ cu.cm} = 1 \text{ litre})$$

Fig. 8.37

Thus, the capacity of the bucket = 26.994 litres.

8.3.4 Volume of a Sphere

(i) Volume of a Solid Sphere

The following simple experiment justifies the formula for volume of a sphere,

$$V = \frac{4}{3}\pi r^3 \text{ cu.units.}$$

Activity

Take a cylindrical shaped container of radius R and height H. Fill the container with water. Immerse a solid sphere of radius r, where $R > r$, in the container and fill the displaced water into another cylindrical shaped container of radius r and height H. The height of the water level is equal to $\frac{4}{3}$ times of its radius ($h = \frac{4}{3}r$). Now, the volume of the solid sphere is same as that of the displaced water.

Volume of the displaced water, V = Base area x Height

$= \pi r^2 \times \frac{4}{3}r$ (here, height of the water level $h = \frac{4}{3}r$)

$= \frac{4}{3}\pi r^3$

Thus, the volume of the sphere, $V = \frac{4}{3}\pi r^3$ cu.units.

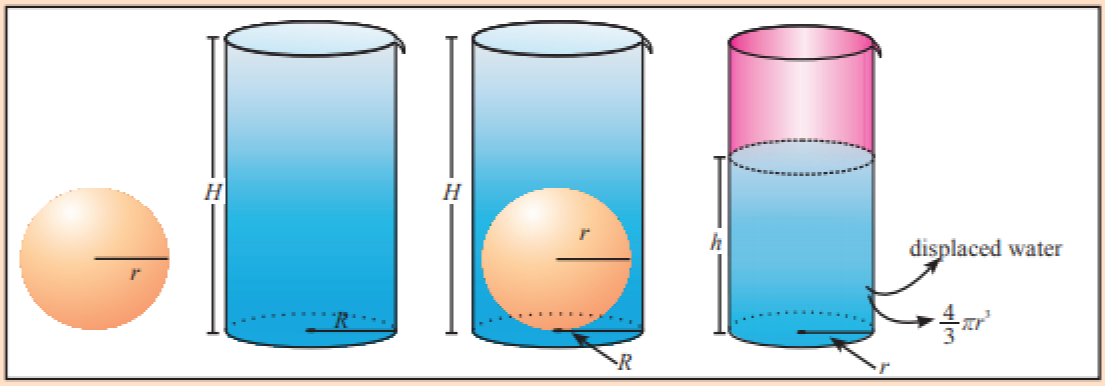


Fig. 8.38

(ii) Volume of a hollow sphere (Volume of the material used)

If the inner and outer radius of a hollow sphere are r and R respectively, then

$$\left.\begin{array}{l}\text{Volume of the}\\ \text{hollow sphere}\end{array}\right\} = \left.\begin{array}{l}\text{Volume of the}\\ \text{outer sphere}\end{array}\right\} - \left\{\begin{array}{l}\text{Volume of the}\\ \text{inner sphere}\end{array}\right.$$

$$= \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$$

$\therefore$ Volume of hollow sphere $= \frac{4}{3}\pi(R^3 - r^3)$ cu. units.

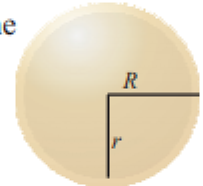


Fig. 8.39

(iii) Volume of a solid hemisphere

Volume of the solid hemisphere $= \frac{1}{2} \times$ volume of the sphere

$$= \frac{1}{2} \times \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}\pi r^3 \text{ cu.units.}$$

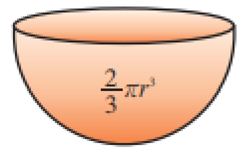


Fig. 8.40

(iv) Volume of a hollow hemisphere (Volume of the material used)

$$\left.\begin{array}{l}\text{Volume of a hollow}\\ \text{hemisphere}\end{array}\right\} = \left.\begin{array}{l}\text{Volume of outer}\\ \text{hemisphere}\end{array}\right\} - \left\{\begin{array}{l}\text{Volume of inner}\\ \text{hemisphere}\end{array}\right.$$

$$= \frac{2}{3} \times \pi \times R^3 - \frac{2}{3} \times \pi \times r^3$$

$$= \frac{2}{3}\pi(R^3 - r^3) \text{ cu.units.}$$

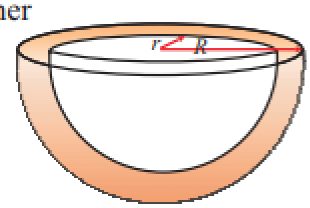


Fig. 8.41

Example 8.17

Find the volume of a sphere-shaped metallic shot-put having diameter of 8.4 cm. (Take $\pi = \frac{22}{7}$)

Solution Let r be radius of the metallic shot-put.

Now, $2r = 8.4$ cm $\implies$ $r = 4.2$ cm

Volume of the shot-put, $V = \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times \frac{42}{10} \times \frac{42}{10} \times \frac{42}{10}$$

Thus, the volume of the shot-put $= 310.464$ cu.cm.

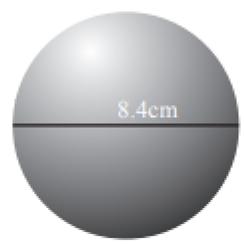


Fig. 8.42

Example 8.18

A cone, a hemisphere and cylinder have equal bases. If the heights of the cone and a cylinder are equal and are same as the common radius, then find the ratio of their respective volumes.

Solution Let r be the common radius of the cone, hemisphere and cylinder.

Let h be the common height of the cone and cylinder.

Given that $r = h$

Let V_1, V_2 and V_3 be the volumes of the cone, hemisphere and cylinder respectively.

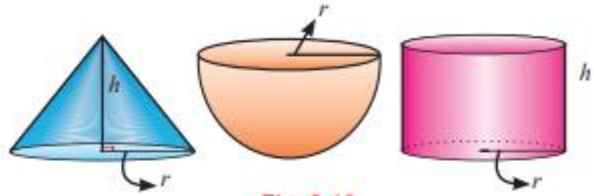


Fig. 8.43

$$\text{Now, } V_1 : V_2 : V_3 = \frac{1}{3}\pi r^2 h : \frac{2}{3}\pi r^3 : \pi r^2 h$$

$$\Longrightarrow \quad = \frac{1}{3}\pi r^3 : \frac{2}{3}\pi r^3 : \pi r^3 \qquad (\text{here, } r = h)$$

$$\Longrightarrow V_1 : V_2 : V_3 = \frac{1}{3} : \frac{2}{3} : 1$$

Hence, the required ratio is $1 : 2 : 3$.

Example 8.19

If the volume of a solid sphere is $7241\frac{1}{7}$ cu.cm, then find its radius. (Take $\pi = \frac{22}{7}$)

Solution Let r and V be the radius and volume of the solid sphere respectively.

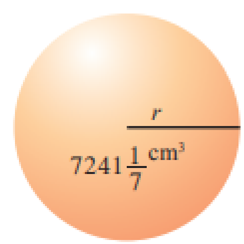


Fig. 8.44

$$\text{Given that} \quad V = 7241\frac{1}{7} \text{ cu.cm}$$

$$\Longrightarrow \quad \frac{4}{3}\pi r^3 = \frac{50688}{7}$$

$$\Longrightarrow \quad \frac{4}{3} \times \frac{22}{7} \times r^3 = \frac{50688}{7}$$

$$r^3 = \frac{50688}{7} \times \frac{3 \times 7}{4 \times 22}$$

$$= 1728 = 4^3 \times 3^3$$

Thus, the radius of the sphere, $r = 12$ cm.

Example 8.20

Volume of a hollow sphere is $\frac{11352}{7}$ cm^3. If the outer radius is 8 cm, find the inner radius of the sphere. (Take $\pi = \frac{22}{7}$)

Solution Let R and r be the outer and inner radii of the hollow sphere respectively.

Let V be the volume of the hollow sphere.

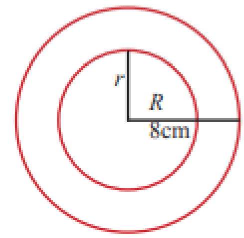


Fig. 8.45

Now, given that $V = \frac{11352}{7}$ cm^3

$$\Longrightarrow \quad \frac{4}{3}\pi(R^3 - r^3) = \frac{11352}{7}$$

$$\Longrightarrow \quad \frac{4}{3}\times\frac{22}{7}(8^3 - r^3) = \frac{11352}{7}$$

$$512 - r^3 = 387 \Longrightarrow r^3 = 125 = 5^3$$

Hence, the inner radius, $r = 5$ cm.

Example 8.21

A solid wooden toy is in the form of a cone surmounted on a hemisphere. If the radii of the hemisphere and the base of the cone are 3.5 cm each and the total height of the toy is 17.5 cm, then find the volume of wood used in the toy. (Take $\pi = \frac{22}{7}$)

Solution Hemispherical portion :

Radius, $r = 3.5$ cm

Conical portion :

Radius, $r = 3.5$ cm

Height, $h = 17.5 - 3.5 = 14$ cm

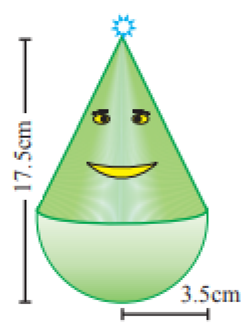


Fig. 8.47

Volume of the wood = Volume of the hemisphere + Volume of the cone

$$= \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$$

$$= \frac{\pi r^2}{3}(2r + h)$$

$$= \frac{22}{7}\times\frac{3.5\times3.5}{3}\times(2\times3.5 + 14) = 269.5$$

Hence, the volume of the wood used in the toy = 269.5 cu.cm.

Example 8.22

A cup is in the form of a hemisphere surmounted by a cylinder. The height of the cylindrical portion is 8 cm and the total height of the cup is 11.5 cm. Find the total surface area of the cup. (Take $\pi = \frac{22}{7}$)

Solution

Hemispherical portion

Radius, r = Total height – 8

$\Longrightarrow r = 11.5 - 8 = 3.5$ cm

Cylindrical portion

Height, $h = 8$ cm.

Thus, radius $r = 3.5 \text{ cm} = \frac{7}{2}$ cm

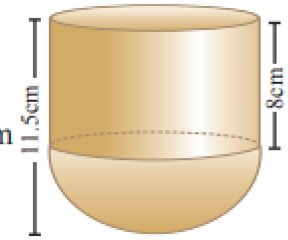


Fig. 8.48

Total surface area of the cup = $\left\{\begin{array}{l}\text{CSA of the hemispherical portion} \\ + \text{ CSA of the cylindrical portion}\end{array}\right.$

$= 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$

$= 2 \times \frac{22}{7} \times \frac{7}{2}\left(\frac{7}{2} + 8\right)$

$\therefore$ Total surface area of the cup = 253 sq.cm.

Example 8.23

A circus tent is to be erected in the form of a cone surmounted on a cylinder. The total height of the tent is 49 m. Diameter of the base is 42 m and height of the cylinder is 21 m. Find the cost of canvas needed to make the tent, if the cost of canvas is ₹12.50/m^2. (Take $\pi = \frac{22}{7}$)

Solution

Cylindrical Part

Diameter, $2r = 42$ m

Radius, $r = 21$ m

Height, $h = 21$ m

Conical Part

Radius, $r = 21$ m

Height, $h_1 = 49 - 21 = 28$ m

Slant height, $l = \sqrt{h_1^2 + r^2}$

$= \sqrt{28^2 + 21^2}$

$= 7\sqrt{4^2 + 3^2} = 35$ m

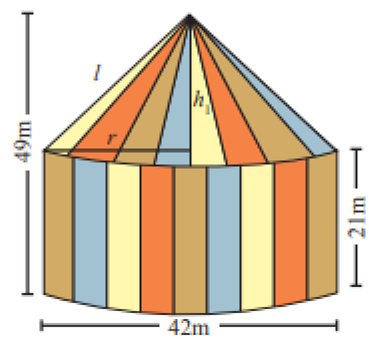


Fig. 8.49

Total area of the canvas needed = CSA of the cylindrical part + CSA of the conical part

$= 2\pi rh + \pi rl = \pi r(2h + l)$

$= \frac{22}{7} \times 21(2 \times 21 + 35) = 5082$

Therefore, area of the canvas = 5082 m^2

Now, the cost of the canvas per sq.m = ₹12.50

Thus, the total cost of the canvas = 5082 × 12.5 = ₹63525.

Example 8.24

A hollow sphere of external and internal diameters of 8 cm and 4 cm respectively is melted and made into another solid in the shape of a right circular cone of base diameter of 8 cm. Find the height of the cone.

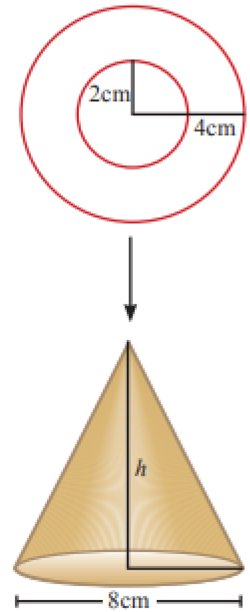


Fig. 8.50

Solution Let R and r be the external and internal radii of the hollow sphere.

Let h and r_1 be the height and the radius of the cone to be made.

Hollow Sphere

External	Internal	Cone
$2R = 8$ cm	$2r = 4$ cm	$2r_1 = 8$
$\Longrightarrow R = 4$ cm	$\Longrightarrow r = 2$ cm	$\Longrightarrow r_1 = 4$

When the hollow sphere is melted and made into a solid cone, we have

$$\text{Volume of the cone} = \text{Volume of the hollow sphere}$$

$$\Longrightarrow \frac{1}{3}\pi r_1^2 h = \frac{4}{3}\pi[R^3 - r^3]$$

$$\Longrightarrow \frac{1}{3}\times\pi\times 4^2\times h = \frac{4}{3}\times\pi\times(4^3 - 2^3)$$

$$\Longrightarrow h = \frac{64-8}{4} = 14$$

Hence, the height of the cone $h = 14$ cm.

Example 8.25

Spherical shaped marbles of diameter 1.4 cm each, are dropped into a cylindrical beaker of diameter 7 cm containing some water. Find the number of marbles that should be dropped into the beaker so that the water level rises by 5.6 cm.

Solution Let n be the number of marbles needed. Let r_1 and r_2 be the radii of the marbles and cylindrical beaker respectively.

Marbles	Cylindrical Beaker
Diameter, $2r_1 = 1.4$ cm	Diameter, $2r_2 = 7$ cm
Radius $r_1 = 0.7$ cm	Radius, $r_2 = \frac{7}{2}$ cm

Let h be the height of the water level raised.

Then, $h = 5.6$ cm

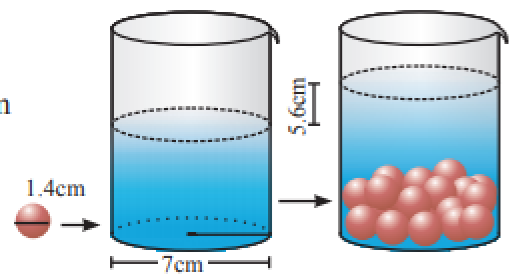


Fig. 8.51

After the marbles are dropped into the beaker,

Volume of water raised = Volume of n marbles

$$\Longrightarrow \pi r_2^2 h = n \times \frac{4}{3}\pi r_1^3$$

Thus, $$n = \frac{3r_2^2 h}{4r_1^3}$$

$$n = \frac{3 \times \frac{7}{2} \times \frac{7}{2} \times 5.6}{4 \times \frac{7}{10} \times \frac{7}{10} \times \frac{7}{10}} = 150.$$

$\therefore$ The number of marbles needed is 150.

Example 8.26

Water is flowing at the rate of 15 km / hr through a cylindrical pipe of diameter 14 cm into a rectangular tank which is 50 m long and 44 m wide. In how many hours will the water level in the tank raise by 21 cm? (Take $\pi = \frac{22}{7}$)

Solution Speed of water = 15 km / hr

= 15000 m / hr

Diameter of the pipe, $2r = 14$ cm

Thus, $r = \frac{7}{100}$ m.

Let h be the water level to be raised.

Thus, $h = 21 \text{ cm} = \frac{21}{100}$ m

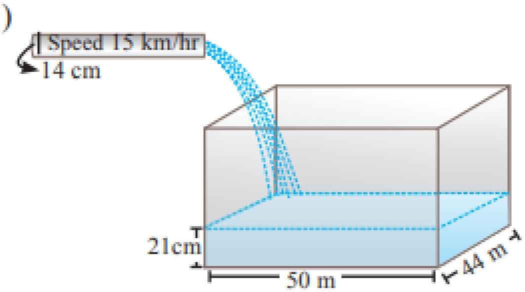


Fig. 8.52

Now, the volume of water discharged

= Cross section area of the pipe × Time × Speed

Volume of water discharged in one hour

$$= \pi r^2 \times 1 \times 15000$$

$$= \frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 15000 \text{ cu.m}$$

Volume of required quantity of water in the tank is,

$$lbh = 50 \times 44 \times \frac{21}{100}$$

Assume that T hours are needed to get the required quantity of water.

$$\therefore \quad \left.\begin{array}{r}\text{Volume of water discharged}\\ \text{in T hours}\end{array}\right\} = \text{Required quantity of water in the tank}$$

$$\Longrightarrow \quad \frac{22}{7} \times \left(\frac{7}{100}\right)^2 \times T \times 15000 = 50 \times 44 \times \frac{21}{100}$$

Thus, $T = 2$ hours.

Hence, it will take 2 hours to raise the required water level.

Example 8.27

A cuboid shaped slab of iron whose dimensions are 55 cm×40 cm×15 cm is melted and recast into a pipe. The outer diameter and thickness of the pipe are 8 cm and 1 cm respectively. Find the length of the pipe. (Take $\pi = \frac{22}{7}$)

Solution Let h_1 be the length of the pipe.

Let R and r be the outer and inner radii of the pipe respectively.

Iron slab: Let $lbh = 55 \times 40 \times 15$.

Iron pipe:

Outer diameter, $2R = 8$ cm

$\therefore$ Outer radius, $R = 4$ cm

Thickness, $w = 1$ cm

$\therefore$ Inner radius, $r = R - w = 4 - 1 = 3$ cm

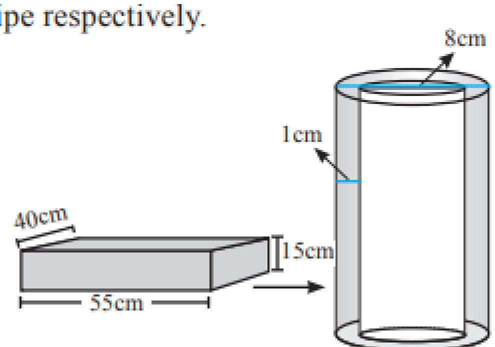


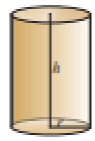
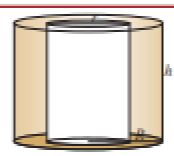
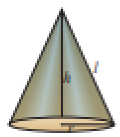
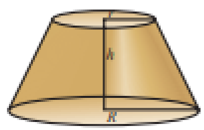
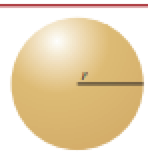
Fig. 8.53

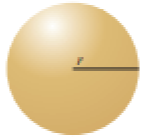
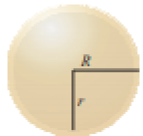
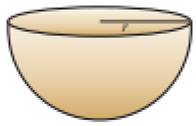
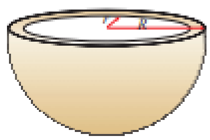
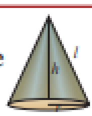
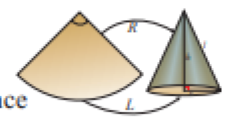
Now, the volume of the iron pipe = Volume of iron slab

$$\Longrightarrow \quad \pi h_1 (R + r)(R - r) = lbh$$

That is, $\frac{22}{7} \times h_1 (4 + 3)(4 - 3) = 55 \times 40 \times 15$

Thus, the length of the pipe, $h_1 = 1500$ cm $= 15$ m.

Sl. No	Name	Figure	Lateral or Curved Surface Area (sq.units)	Total Surface Area (sq.units)	Volume (cu.units)
1	**Solid right circular cylinder**		$2\pi rh$	$2\pi r(h+r)$	$\pi r^2 h$
2	**Right circular hollow cylinder**		$2\pi h(R+r)$	$2\pi(R+r)(R-r+h)$	Volume of the material used $\pi R^2 h - \pi r^2 h$ $= \pi h(R^2 - r^2)$ $= \pi h(R+r)(R-r)$
3	**Solid right circular cone**		πrl	$\pi r(l+r)$	$\frac{1}{3}\pi r^2 h$
4	**Frustum**		- - - - - - -	- - - - - - - - - - - - - - -	$\frac{1}{3}\pi h(R^2+r^2+Rr)$
5	**Sphere**		$4\pi r^2$	- - -	$\frac{4}{3}\pi r^3$

5	**Sphere**		$4\pi r^2$	- - -	$\frac{4}{3}\pi r^3$
6	**Hollow sphere**		- - -	- - -	Volume of the material used $\frac{4}{3}\pi(R^3 - r^3)$
7	**Solid Hemisphere**		$2\pi r^2$	$3\pi r^2$	$\frac{2}{3}\pi r^3$
8	**Hollow Hemisphere**		$2\pi(R^2 + r^2)$	$2\pi(R^2 + r^2) + \pi(R^2 - r^2)$ $= \pi(3R^2 + r^2)$	Volume of the material used $\frac{2}{3}\pi(R^3 - r^3)$
9	**A sector of a circle converted into a Cone**  CSA of a cone = Area of the sector $\pi rl = \frac{\theta}{360} \times \pi r^2$ Length of the sector = Base circumference of the cone		$l = \sqrt{h^2 + r^2}$ $h = \sqrt{l^2 - r^2}$ $r = \sqrt{l^2 - h^2}$	10. Volume of water flows out through a pipe = {Cross section area × Speed × Time } 11. No. of new solids obtained by recasting $= \dfrac{\text{Volume of the solid which is melted}}{\text{volume of one solid which is made}}$	
12	Conversions	1 m³ = 1000 litres , 1 d.m³ = 1 litre ,	1000 cm³ = 1 litre ,	1000 litres = 1 kl	